Mathematical Finance, Vol. 17, No. 2 (April 2007), 307–317

STOCK LOANS

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This paper introduces a mathematical model for a currently popular financial product called a stock loan. Quantitative analysis is carried out to establish explicitly the value of such a loan, as well as the ranges of fair values of the loan size and interest, and the fee for providing such a service.

KEY WORDS: stock loan, Black–Scholes model, call option, stopping time

1. INTRODUCTION

We consider a simple economy where a client (borrower), who owns one share of a stock, obtains a loan from a bank (lender) with the share as collateral. The client may regain the stock on or before, depending on the type of the loan, the loan maturity by repaying the bank the principal and interest, or surrender the stock instead of repaying the loan. Such a loan is called a *stock loan* or *security loan*, which is currently a very popular service provided by many banks and financial firms.[1]

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From another perspective, for investors who do not have equity positions yet with limited funds, stock loan may serve as a leverage device for them to take advantage of the potential stock rise. In such a case a stock loan is more like a real estate mortgage (although due to less liquidity the price movement of a house behaves quite differently from that of a stock).

A natural problem arises for both the client and the bank: what are the fair values of the principal, the loan interest, and the fee charged by the bank for providing the service? To the authors' best knowledge, few results on this problem have been reported in the literature. The aim of this paper is to provide a complete quantitative analysis of this problem. In the next section, we formulate a mathematical model of the stock loan and show that it is essentially an American call option with a *time-dependent* strike price or, equivalently, one with a possibly *negative* interest rate. In Section 3, we provide an explicit formula for the value of the stock loan. In Section 4, we apply the results in Section 3 to work out the rational values of the parameters. Concluding remarks are given in Section 5.

2. PROBLEM FORMULATION

We consider the standard Black–Scholes model in a continuous-time financial market consisting of two assets: a bond and a stock. The continuously compounding interest rate of the bond is assumed to be a constant $r > 0$. The uncertainty associated with the stock is described by a filtered risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which a standard Brownian motion $W \equiv \{W_t,\ t \geq 0\}$ is defined, where $(\mathcal{F}_t)_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by W, with $\mathcal{F}_0 = \sigma\{\emptyset, \Omega\}$ and $\mathcal{F} = \sigma\{\bigcup_{t\geq 0} \mathcal{F}_t\}$. The market price process of the stock follows a geometric Brownian motion

$$S_t = S_0 \exp\left\{(r-\delta)t + \sigma W_t - \frac{\sigma^2}{2} t\right\}, \quad t \geq 0, \tag{2.1}$$

where $S_0 > 0$ is the initial price of the stock, $\sigma > 0$ is the volatility, and $\delta \geq 0$ is the dividend yield.

A stock loan model under consideration in this paper has the following specifications:

- At time 0, a client borrows amount q ($q > 0$) from a bank with one share of the stock as collateral, whereas the bank charges amount c ($0 \leq c \leq q$) for the service. As a consequence, the client gets amount $(q - c)$ from the bank.
- The continuously compounding loan interest rate is γ. The client may regain the stock by repaying amount $qe^{\gamma t}$ to the bank at any time $t \geq 0$. The dividends of the stock accrued are collected by the bank until the client regains the stock, and the paid dividends are not credited to the client.
- The client is not obliged to regain the stock.

The question is: what are the rational values of the parameters q, c, and γ?

The above problem can be regarded as the client initially buying at price $(S_0 - q + c)$ an American option with a payoff process $Y_t = (S_t - qe^{\gamma t})_+$, $t \geq 0$, where $a_+ := \max\{a, 0\}$ for any real number a. We call the value of this option the (initial) *value* or *price* of the underlying stock loan. The rational values of q, c, and γ should be such that the value of the stock loan is $(S_0 - q + c)$. The crucial difference between this option and the conventional American option is that the former has a time-dependent strike price. Thus our problem is essentially to evaluate an American call option with a time-dependent strike price. It should be emphasized that an American option with a time-dependent strike price is not a straightforward adaptation of its constant-strike counterpart, and there is an inherent technical subtlety associated with our problem (see discussion below and Remark 3.1). This is also attested by the fact that our stock loan value function is structurally different from that of a conventional perpetual American option (see Remark 3.2).

One may argue that one could use a simple transformation to turn the problem into one with a time-independent strike price. Specifically, the initial value function of the option with respect to the initial stock price $\mathbf{S}_0 = x$ is

$$f(x) := \sup_{\tau\in\mathcal{T}_0} \mathbb{E}\big[e^{-r\tau}(\mathbf{S}_\tau - qe^{\gamma\tau})_+\big] = \sup_{\tau\in\mathcal{T}_0} \mathbb{E}\big[e^{-(r-\gamma)\tau}(\tilde{\mathbf{S}}_\tau - q)_+\big],$$

where $\mathcal{T}_0$ denotes all $(\mathcal{F}_t)_{t\geq 0}$-stopping times, and $\tilde{\mathbf{S}}_t \equiv e^{-\gamma t}\mathbf{S}_t$ is given by

$$\tilde{\mathbf{S}}_t = x\exp\left\{(r-\gamma-\delta)t + \sigma W_t - \frac{\sigma^2}{2}t\right\}, \quad t \geq 0.$$

In other words, our problem is also equivalent to a *conventional* perpetual American call option with a possibly *negative* interest rate $\tilde{r} := r - \gamma$ (because in the context of our model, the loan rate γ is usually larger than the risk-free rate r). This negative interest rate $\tilde{r}$ leads to a major difficulty in applying the standard approach involving a variational inequality and the smooth-fit principle (see, e.g., Karatzas and Shreve 1998, pp. 60–67) to solve the problem. To elaborate, assume $\delta = 0$ for simplicity. The variational inequality that f must satisfy is (cf. Karatzas and Shreve 1998, p. 64)

$$\begin{cases} \max\left\{\frac{1}{2}\sigma^2x^2f'' + \tilde{r}xf' - \tilde{r}f,\ (x-q)_+ - f\right\} = 0,\ x > 0, \\ f(0) = 0. \end{cases} \tag{2.2}$$

One should then solve the equation

$$\frac{1}{2}\sigma^2x^2f'' + \tilde{r}xf' - \tilde{r}f = 0 \tag{2.3}$$

on a certain interval $[0, b)$ (the so-called continuation region) and smoothly fit with the solution $f(x) = (x - q)$ on (b, ∞) (the stopping region). The point $b \in [0, \infty]$ and other related unknown coefficients will be determined based on the smooth fit at b. Specifically, the general solution to (2.3) is

$$f(x) = C_1x^{\lambda_1} + C_2x^{\lambda_2},$$

where $\lambda_{1,2}$ are the solutions to the indicial equation

$$\frac{1}{2}\sigma^2\lambda^2 + \left(\tilde{r} - \frac{1}{2}\sigma^2\right)\lambda - \tilde{r} = 0, \tag{2.4}$$

which has two solutions

$$\lambda_1 = \frac{\left(-\tilde{r} + \frac{1}{2}\sigma^2\right) + \left|\tilde{r} + \frac{1}{2}\sigma^2\right|}{\sigma^2}, \quad \lambda_2 = \frac{\left(-\tilde{r} + \frac{1}{2}\sigma^2\right) - \left|\tilde{r} + \frac{1}{2}\sigma^2\right|}{\sigma^2}.$$

If $\tilde{r} > 0$

or $C_1 = 1,\ b = +\infty$. Therefore, $f(x) = x\ \forall x \geq 0$, recovering the familiar result that the initial value of a perpetual American call option (without dividend) is the initial stock price.

Now, if $\tilde{r}$ is negative (the present case) with $\tilde{r} + \frac{1}{2}\sigma^2 < 0$, then $\lambda_1 = \frac{-2\tilde{r}}{\sigma^2} > 1$, $\lambda_2 = 1$. As opposed to the conventional case, none of the λ_i's will be rejected by the initial condition $f(0) = 0$. Thus $f(x) = C_1 x^{\lambda_1} + C_2 x$, and we have *three* unknown parameters (C_1, C_2, and b) while there are only *two* equations based on the smooth fit at b. This explains the major difficulty in using the variational inequality approach. In this paper, we choose to use a pure probabilistic approach to solve our problem.

The above analysis also shows that (1) the American call option pricing with a *negative* interest rate is a meaningful problem, and (2) the problem cannot be solved directly by a variational inequality (or Black–Scholes) approach.

To conclude this section, notice that our stock loan model is structured with an infinite life. Practically, the maturity of the loan is finite (although many such products do allow renewal or refinancing for a subsequent term). However, for mathematical tractability, we assume for now that the maturity of the loan is infinite; thereby we are dealing with a perpetual American option. It remains a challenging open problem to fully analyze a finite-term American call option with time-dependent strike prices and dividend payments.

3. STOCK LOAN EVALUATION

In this section, we compute the value of the stock loan, or that of a perpetual American option with a payoff process $Y_t = (\mathbf{S}_t - qe^{\gamma t})_+$, $t \geq 0$. Note that since the payoff process of the stock loan $Y_t \geq 0$ a.s., and $Y_t > 0$ with a positive probability, to avoid arbitrage we must have the following standing assumption:

STANDING ASSUMPTION. $\mathbf{S}_0 - q + c > 0$.

Now, by the law of the iterated logarithm for Brownian motion, we see that $\limsup_{t\to\infty}(e^{-rt}Y_t) = 0$ a.s. So we define $(e^{-rt}Y_t)|_{t=+\infty} := 0$.

The value of the American option at time t is (cf. Shiryaev et al. 1994)

$$V_t = \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\big(\mathbf{S}_\tau - qe^{\gamma\tau}\big)_+ \big|\mathcal{F}_t\right], \tag{3.1}$$

where $\mathcal{T}_t$ denotes all $(\mathcal{F}_s)_{s\geq 0}$-stopping times τ with $\tau \geq t$ a.s. In particular, the initial value function of the option with respect to the initial stock price x is

$$f(x) := \sup_{\tau\in\mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(xe^{(r-\delta)\tau+\sigma W_\tau - \frac{\sigma^2}{2}\tau} - qe^{\gamma\tau}\right)_+\right]. \tag{3.2}$$

Next, we introduce some qualitative properties of the value function f, which are helpful in solving the optimal stopping time problem (3.2).

PROPOSITION 3.1. *f is convex, continuous and nondecreasing on* (0, where the last inequality follows from a standard argument involving the nonnegative martingale property of $\{e^{\sigma W_t - \frac{\sigma^2}{2}t}, t \geq 0\}$, the optional sampling theorem and Fatou's lemma. Finally, by its definition f is obviously convex and nondecreasing. Since f is finite on its domain, the convexity implies the continuity. □

COROLLARY 3.1. *Let* $k = \inf\{x > 0 : x - q \geq f(x)\}$, *where* $\inf \emptyset := \infty$. *Then* $k \geq q$ *and*

$$\{x > 0 : x - q \geq f(x)\} = [k, \infty).$$

Proof. It is clear for the case with $k = \infty$. Now we suppose $k \in [q, \infty)$, then we have $f(k) = k - q$. We claim that $f(x) = x - q$ for all $x \geq k$, which implies the conclusion of the corollary. Otherwise, by Proposition 3.1, there exists a $k_0 \in (k, \infty)$ such that $f(k_0) > k_0 - q$. Then we have $\beta := \frac{f(k_0) - f(k)}{k_0 - k} > 1$. By the convexity of $f(x)$, we have

$$\frac{f(x) - f(k)}{x - k} \geq \frac{f(k_0) - f(k)}{k_0 - k} = \beta \qquad \text{for all } x \geq k_0,$$

or

$$f(x) \geq \beta(x - k) + k - q \qquad \text{for all } x \geq k_0,$$

which implies $f(x) > x$ for sufficiently large x. Thus we arrive at a contradiction to Proposition 3.1. □

Now we consider a stopping time defined as

$$\tau^* := \inf\{t \geq 0 : S_t - qe^{\gamma t} \geq V_t\},$$

which will be shown to be optimal for problem (3.2) (see Proposition 3.3 below).

PROPOSITION 3.2. *The stopping time* τ^* *has the form*

$$\tau^* \equiv \tau_a = \inf\{t \geq 0 : e^{-\gamma t} S_t = a\} \tag{3.3}$$

for some $a \geq q \vee S_0 := \max\{q, S_0\}$.

Proof. Substituting (2.1) into (3.1), we have

$$\begin{aligned} V_t &= e^{\gamma t} \cdot \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau - t)}\left(e^{-\gamma t} S_t e^{(r-\delta)(\tau - t) + \sigma(W_\tau - W_t) - \frac{\sigma^2}{2}(\tau - t)} - qe^{\gamma(\tau - t)}\right)_+ \Big| \mathcal{F}_t\right] \\ &= e^{\gamma t} \cdot \operatorname*{esssup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(xe^{(r-\delta)\tau + \sigma W_\tau - \frac{\sigma^2}{2}\tau} - qe^{\gamma \tau}\right)_+\right]_{x = e^{-\gamma t} S_t} \\ &= e^{\gamma t} f\big(e^{-\gamma t} S_t\big). \end{aligned}$$

Thus the stopping time

$$\begin{aligned} \tau^* &= \inf\{t \geq 0 : S_t - qe^{\gamma t} \geq e^{\gamma t} f(e^{-\gamma t} S_t)\} \\ &= \inf\{t \geq 0 : e^{-\gamma t} S_t \geq k\}, \end{aligned} \tag{3.4}$$

where k is defined in Corollary 3.1. Let k ($k \geq q$) be defined as in Corollary 3.1. If $S_0 < k$, then by continuity of S_t and (3.4) we have $\tau^* = \tau_k$. If $S_0 \geq k$, then $\tau^* = 0 = \tau_{S_0}$. □

We need to have the following technical result.

LEMMA 3.1. *If $\delta > 0$, or $\delta = 0$ and $\gamma - r > \frac{\sigma^2}{2}$, then*

$$\mathbb{E}\left[\sup_{t\geq 0} e^{-rt}\big(\mathbf{S}_t - qe^{\gamma t}\big)_+\right] < \infty. \tag{3.5}$$

Proof. For $\lambda > 1$ and $\eta > 1$ satisfying $\frac{1}{\lambda} + \frac{1}{\eta} = 1$, and any sufficiently large $z > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\geq 0} e^{-rt}\big(\mathbf{S}_t - qe^{\gamma t}\big)_+ > z\right)\\
&= \mathbb{P}\big(\exists t \geq 0, e^{-rt}\big(\mathbf{S}_t - qe^{\gamma t}\big) > z\big)\\
&= \mathbb{P}\left(\exists t \geq 0,\ W_t - \left(\frac{\sigma}{2} + \frac{\delta}{\sigma}\right)t > \frac{1}{\sigma}\log\left(\frac{z}{\mathbf{S}_0} + \frac{q}{\mathbf{S}_0}e^{(\gamma-r)t}\right)\right)\\
&\leq \mathbb{P}\left(\exists t \geq 0,\ W_t - \left(\frac{\sigma}{2} + \frac{\delta}{\sigma}\right)t > \frac{1}{\lambda\sigma}\log\frac{\lambda z}{\mathbf{S}_0} + \frac{1}{\eta\sigma}\log\left(\frac{\eta q}{\mathbf{S}_0}e^{(\gamma-r)t}\right)\right) \qquad (3.6)\\
&= \mathbb{P}\left(\exists t \geq 0,\ W_t - \left(\frac{\sigma}{2} + \frac{\delta}{\sigma} + \frac{\gamma - r}{\eta\sigma}\right)t > \frac{1}{\lambda\sigma}\log\frac{\lambda z}{\mathbf{S}_0} + \frac{1}{\eta\sigma}\log\frac{\eta q}{\mathbf{S}_0}\right)\\
&= \mathbb{P}\left(\sup_{0\leq t<\infty}\left(W_t - \left(\frac{\sigma}{2} + \frac{\delta}{\sigma} + \frac{\gamma - r}{\eta\sigma}\right)t\right) > \frac{1}{\lambda\sigma}\log\frac{\lambda z}{\mathbf{S}_0} + \frac{1}{\eta\sigma}\log\frac{\eta q}{\mathbf{S}_0}\right)
\end{aligned}$$

$$= \exp\left\{-2\left(\frac{\sigma}{2} + \frac{\delta}{\sigma} + \frac{\gamma - r}{\eta\sigma}\right)\cdot\left(\frac{1}{\lambda\sigma}\log\frac{\lambda z}{\mathbf{S}_0} + \frac{1}{\eta\sigma}\log\frac{\eta q}{\mathbf{S}_0}\right)\right\} \tag{3.7}$$

$$= \left(\frac{\eta q}{\mathbf{S}_0}\right)^{-\left(\frac{2\delta}{\eta\sigma^2} + \frac{1}{\eta} + \frac{2(\gamma-r)}{\eta^2\sigma^2}\right)}\cdot\left(\frac{\lambda z}{\mathbf{S}_0}\right)^{-\left(\frac{2\delta}{\lambda\sigma^2} + \frac{1}{\lambda} + \frac{2(\gamma-r)}{\lambda\eta\sigma^2}\right)}, \tag{3.8}$$

where (3.6) follows from the concavity of the logarithm function and (3.7) follows from a well-known result on the distribution of Brownian functional (see, e.g., Borodin and Salminen 2002, p. 251). It is clear that

$$\frac{2\delta}{\lambda\sigma^2} + \frac{1}{\lambda} + \frac{2(\gamma - r)}{\lambda\eta\sigma^2} = \frac{2(\eta - 1)\delta}{\eta\sigma^2} + \frac{\sigma^2\eta^2 + (2(\gamma - r) - \sigma^2)\eta - 2(\gamma - r)}{\sigma^2\eta^2} > 1 \tag{3.9}$$

for any fixed, sufficiently large η since we have either $\delta > 0$, or $\delta = 0$ and $\gamma - r > \frac{\sigma^2}{2}$. Consequently, (3.5) follows from (3.8) and (3.9). □

PROPOSITION 3.3. *Under the condition of Lemma 3.1, τ^* solves the optimal stopping problem in (3.2) with $x = \mathbf{S}_0$.*

Proof. By Lemma 3.1, it can be proved that the stopping time defined as follows

$$\tau_* = \inf\left\{t \geq 0 : \big(\mathbf{S}_t - qe^{\gamma t}\big)_+ \geq V_t\right\}$$

solves the optimal stopping problem in (3.2) with $x = \mathbf{S}_0$. It is clear that $\tau_* \leq \tau^*$ a.s. On the other hand, whenever $\mathbf{S}_{\tau_*} \geq qe^{\gamma\tau_*}$ we have

$$\mathbf{S}_{\tau_*} - qe^{\gamma\tau_*} = (\mathbf{S}_{\tau_*} - qe^{\gamma\tau_*})_+ = V_{\tau_*}$$

and by definition of τ^*, $\tau^* \le \tau_*$ a.s. Hence we have proved that $[\mathbf{S}_{\tau_*} \ge q e^{\gamma \tau_*}] \subset [\tau_* = \tau^*]$. Accordingly,

$$\begin{aligned} e^{-r\tau_*}\left(\mathbf{S}_{\tau_*} - q e^{\gamma\tau_*}\right)_+ &= e^{-r\tau_*}\left(\mathbf{S}_{\tau_*} - q e^{\gamma\tau_*}\right) 1_{[\mathbf{S}_{\tau_*} \ge q e^{\gamma\tau_*}]} \\ &= e^{-r\tau^*}\left(\mathbf{S}_{\tau^*} - q e^{\gamma\tau^*}\right) 1_{[\mathbf{S}_{\tau_*} \ge q e^{\gamma\tau_*}]} \\ &\le e^{-r\tau^*}\left(\mathbf{S}_{\tau^*} - q e^{\gamma\tau^*}\right) \\ &= e^{-r\tau^*}\left(\mathbf{S}_{\tau^*} - q e^{\gamma\tau^*}\right)_+, \end{aligned}$$

which yields the conclusion of the proposition. □

COROLLARY 3.2. *Under the condition of Lemma 3.1, the initial value of the stock loan is*

$$f(\mathbf{S}_0) = \sup_{a \ge q \vee \mathbf{S}_0} g(a),$$

where

$$g(a) := \mathbb{E}\left[e^{-r\tau_a}\left(\mathbf{S}_{\tau_a} - q e^{\gamma\tau_a}\right)_+\right] = (a - q)\mathbb{E}\left[e^{(\gamma - r)\tau_a} 1_{[\tau_a < \infty]}\right]$$

and τ_a is defined as in (3.3).

In what follows, we will compute $g(a)$. Denote

$$\mu = -\left(\frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}\right), \qquad b = \frac{1}{\sigma} \log \frac{a}{\mathbf{S}_0}. \tag{3.10}$$

Then by (2.1), τ_a can be rewritten as

$$\tau_a = \inf\{t \ge 0 : W_t + \mu t = b\}.$$

The following result, which extends the range of the index λ for Laplace transform of the law of the first hitting time of Brownian motion with drift, is of separate interest.

LEMMA 3.2. *If $\mu^2 - 2\lambda \ge 0$, then*

$$\mathbb{E}\left[e^{\lambda\tau_a} 1_{[\tau_a < \infty]}\right] = e^{\mu b - |b|\sqrt{\mu^2 - 2\lambda}}.$$

Proof. It is a well-known result [see, e.g., Karatzas and Shreve 1991, (5.12) on p. 197] that the density of τ_a is

$$P(\tau_a \in dt) = \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{(b-\mu t)^2}{2t}} dt,\ t > 0.$$

Thus

$$\mathbb{E}\left[e^{\lambda\tau_a} 1_{[\tau_a < \infty]}\right] = \int_0^\infty \frac{|b|}{\sqrt{2\pi t^3}} e^{\lambda t} e^{-\frac{(b-\mu t)^2}{2t}} dt. \tag{3.11}$$

If $\mu^2 - 2\lambda > 0$, then let $\varepsilon > 0$ be small enough such that $\mu^2 - 2(\lambda + \varepsilon) > 0$. We have

$$\begin{aligned} \mathbb{E}\left[e^{\lambda\tau_a} 1_{[\tau_a < \infty]}\right] &= e^{-b\left(\sqrt{\mu^2 - 2(\lambda+\varepsilon)} - \mu\right)} \int_0^\infty \frac{|b|}{\sqrt{2\pi t^3}} e^{-\varepsilon t} e^{-\frac{\left(b - \sqrt{\mu^2 - 2(\lambda+\varepsilon)}\, t\right)^2}{2t}} dt \\ &= e^{-b(\sqrt{\mu^2 - 2(\lambda+\varepsilon)} - \mu)} e^{\sqrt{\mu^2 - 2(\lambda+\varepsilon)}\, b - |b|\sqrt{\mu^2 - 2(\lambda+\varepsilon) + 2\varepsilon}} \\ &= e^{\mu b - |b|\sqrt{\mu^2 - 2\lambda}}, \end{aligned}$$

where the second equality follows a well-known result on the Laplace transform of the law of the hitting time of Brownian motion with drift (see, e.g., Karatzas and Shreve 1991, Exercise 5.10 on p. 197). Finally, for the case when $\mu^2 - 2\lambda = 0$, the conclusion follows from considering a sequence $\lambda_n \uparrow \lambda$ along with the monotone convergence theorem. □

REMARK 3.1. To calculate the left hand side of (3.11) poses the main technical difficulty for pricing an American call option with time-dependent strike prices. This is because in our case, $\lambda = \gamma - r$ may well be positive, whereas in the case of a conventional American option the index λ for the corresponding Laplace transform is automatically negative so that the exponential martingale technique can be applied (see Borodin and Salminen 2002; Karatzas and Shreve 1991). We get around this by imposing a weaker condition $\mu^2 - 2\lambda \geq 0$, which *happens* to be satisfied in our stock loan problem (see below).

By (3.10), it is clear that

$$\mu^2 - 2(\gamma - r) = \left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta \geq 0.$$

Hence it follows from Lemma 3.2 that, for all $a \geq \mathbf{S}_0$,

$$g(a) = (a - q)\left(\frac{a}{\mathbf{S}_0}\right)^{-\frac{1}{\sigma}\left[\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}\right]}. \tag{3.12}$$

Now we can claim the main result of this section.

THEOREM 3.1. *We have the following assertions on* $g(a)$ *and* $f(\mathbf{S}_0)$:

(*a*) *If* $\delta = 0$ *and* $\gamma - r \leq \frac{\sigma^2}{2}$, *then* $g(a) = \frac{(a-q)\mathbf{S}_0}{a}$ *for* $a \geq \mathbf{S}_0$ *and* $f(\mathbf{S}_0) = \mathbf{S}_0$.
(*b*) *If* $\delta > 0$, or $\delta = 0$ *and* $\gamma - r > \frac{\sigma^2}{2}$, *then*

$$a_0 := \frac{q\left[\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}\right]}{\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} - \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}} > q,$$

and we have the following two cases:

(*b1*) *If* $q < a_0 \leq \mathbf{S}_0$, *then* $g(a)$ *attains its maximum on* $[\mathbf{S}_0, \infty)$ *at* $a = \mathbf{S}_0$ *and* $f(\mathbf{S}_0) = \mathbf{S}_0 - q$.
(*b2*) *If* $a_0 > \mathbf{S}_0$, *then* $g(a)$ *attains its maximum on* $[q \vee \mathbf{S}_0, \infty)$ *at* $a = a_0$ *and* $f(\mathbf{S}_0) = g(a_0)$.

Proof. It is clear that

$$\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} - \left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right) \geq 0,$$

where the equality holds if and only if $\delta = 0$ and $\gamma - r \leq \frac{\sigma^2}{2}$.

In Case (a), it is straightforward that $g(a) = \frac{(a-q)\mathbf{S}_0}{a}$ for $a \geq \mathbf{S}_0$. By (3.2), we have $f(\mathbf{S}_0) \geq \sup_{a \geq \mathbf{S}_0} g(a) = \mathbf{S}_0$ and hence by Proposition 3.1, $f(\mathbf{S}_0) = \mathbf{S}_0$.

In Case (b), it is evident that $a_0 > q$. For $a > q \vee \mathbf{S}_0$, we have from (3.12) that

$$\log g(a) = \log(a - q) - \frac{1}{\sigma}\left[\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}\right] \log \frac{a}{\mathbf{S}_0}$$

and

$$\frac{d(\log g(a))}{da}$$

$$= \frac{-\left[\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} - \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}\right] a + q\left[\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}\right]}{\sigma a(a - q)}.$$

The numerator, as a function of a, is decreasing. Since a_0 makes the numerator to be zero, we conclude:

- If $q < a_0 \le \mathbf{S}_0$, then $\frac{d(\log g(a))}{da} < 0$ for $a > \mathbf{S}_0$, which implies that $g(a)$ is decreasing on $[\mathbf{S}_0, \infty)$; hence it attains its maximum at $a = \mathbf{S}_0$. In this case, by Corollary 3.2, $f(\mathbf{S}_0) = g(\mathbf{S}_0) = \mathbf{S}_0 - q$.
- If $a_0 > \mathbf{S}_0$, then $\frac{d(\log g(a))}{da} > 0$ for $\mathbf{S}_0 < a < a_0$ and $\frac{d(\log g(a))}{da} < 0$ for $a > a_0$. Thus in $[q \vee \mathbf{S}_0, \infty)$, $g(a)$ attains its maximum at $a = a_0$ and by Corollary 3.2, $f(\mathbf{S}_0) = g(a_0)$. □

REMARK 3.2. We see from the preceding theorem that a perpetual American option with a time-varying strike price, $qe^{\gamma t}$, indeed has a structurally different value process as compared with the counterpart with a constant strike price. Let us consider the case without dividend ($\delta = 0$). In this case, the value of a conventional perpetual American option with a constant strike q is $f(\mathbf{S}_0) = \mathbf{S}_0$ [see, e.g., Karatzas and Shreve 1998, equation (6.17) on p. 65 with $\delta = 0$]. However, the initial value of our option is

$$f(\mathbf{S}_0) = \begin{cases} \mathbf{S}_0, & \text{if } \gamma \le r + \dfrac{\sigma^2}{2}; \\ \dfrac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha} q^{1-\alpha} \mathbf{S}_0^\alpha, & \text{if } \gamma > r + \dfrac{\sigma^2}{2} \text{ and } q > \dfrac{\alpha - 1}{\alpha} \mathbf{S}_0; \\ \mathbf{S}_0 - q, & \text{if } \gamma > r + \dfrac{\sigma^2}{2} \text{ and } q \le \dfrac{\alpha - 1}{\alpha} \mathbf{S}_0, \end{cases}$$

where $\alpha := \frac{2(\gamma - r)}{\sigma^2} > 1$ if $\gamma > r + \frac{\sigma^2}{2}$. So, when γ is small ($\gamma \le r + \frac{\sigma^2}{2}$), our option has the same value as a perpetual European or American option with a constant strike price. But this does not hold true when γ is large ($\gamma > r + \frac{\sigma^2}{2}$). Incidentally, this also echos the variational inequality analysis in Section 2 where it was shown that a difficulty arises when $\gamma > r + \frac{\sigma^2}{2}$.

4. FAIR VALUES OF THE PARAMETERS

Now, we are in the position to apply Theorem 3.1 to work out the fair values for the parameters γ, q, and c. We proceed for three cases:

Case (a) in Theorem 3.1. This is the case when there is no dividend ($\delta = 0$) and the excess loan interest rate over the risk-free interest rate is small ($\gamma - r \le \frac{\sigma^2}{2}$). By

Theorem 3.1 the initial value of the stock loan is $f(\mathbf{S}_0) = \mathbf{S}_0$. In order to have $f(\mathbf{S}_0) = \mathbf{S}_0 - q + c$, it must be that $q - c = 0$. This means that the loan interest rate is too small for the bank to have an incentive to actually carry out the business (it charges an amount equal to the loan size, effectively giving no money to the client). In this case, the client at the initial time exchanges one share of the stock for a perpetual American option with the payoff process $(\mathbf{S}_t - qe^{\gamma t})_+$ for a notional amount q (the specific value of q is insignificant). This is not an interesting case.

Case (b1) in Theorem 3.1. In this case, the bank would receive dividend ($\delta > 0$) and/or the loan interest rate is sufficiently large ($\gamma - r > \frac{\sigma^2}{2}$), and the loan-to-value is not large enough, i.e.,

$$\frac{q}{\mathbf{S}_0} \leq \frac{\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} - \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}}{\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}} \quad (\leq 1).$$

By Theorem 3.1 the initial value of the stock loan is $f(\mathbf{S}_0) = \mathbf{S}_0 - q$. In order to have $f(\mathbf{S}_0) = \mathbf{S}_0 - q + c$, one has $c = 0$, which means that the bank does not need to charge a fee for the service. As a result, initially the client obtains the stock loan at the price $\mathbf{S}_0 - q$. However, Theorem 3.1 also suggests that the optimal exercise time is $t = 0$; hence there is actually no exchange between the client and the bank (or that there is not enough incentive for the client to do the business). This case is also not interesting.

Case (b2) in Theorem 3.1. In this case, both parties have the incentives to do the business (the bank does since there is dividend payment and/or the loan interest rate is high enough, and so does the client as the loan-to-value is sufficiently high). It follows from Theorem 3.1 that the initial value of the stock loan is $f(\mathbf{S}_0) = g(a_0) > \mathbf{S}_0 - q$. Then the bank can charge an amount $c = g(a_0) - \mathbf{S}_0 + q$ from the client for the service. The fair values of the parameters q, c, and γ should be such that

$$\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} - \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma} > 0$$

$$\left(\text{i.e., } \delta > 0, \text{ or } \gamma - r > \frac{\sigma^2}{2}\right),$$

$$q > \frac{\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} - \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}}{\sqrt{\left(\frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma}\right)^2 + 2\delta} + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}} \mathbf{S}_0,$$

and

$$c = g(a_0) - \mathbf{S}_0 + q.$$

The optimal time for the client to regain the stock is when the stock price discounted to the initial time (using the loan interest rate), i.e., $e^{-\gamma t}\mathbf{S}_t$, hits a_0 for the first time.

In particular, if $\delta = 0$, then the fair values of the parameters q, c, and γ should be such that

$$\gamma > r + \frac{\sigma^2}{2},$$

$$q > \frac{\alpha - 1}{\alpha} S_0$$

and

$$c = \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha} q^{1-\alpha} S_0^\alpha - S_0 + q,$$

where $\alpha := \frac{2(\gamma - r)}{\sigma^2} > 1$. The client should claim the stock back as soon as the present value (evaluated at the loan interest rate) of the stock reaches $a_0 = \frac{\alpha}{\alpha - 1} q$.

EXAMPLE 4.1. Consider a model where $r = 0.05$, $\sigma = 0.15$, $\gamma = 0.07$, $\delta = 0$ and $S_0 = 100$. Then $\alpha = 1.7778 > 1$, and $\frac{\alpha - 1}{\alpha} = 0.4376$. This means any stock loan with a loan-to-value over 43.76% is marketable. The following is a table of service charge versus loan size based on the aforementioned formula.

q	50	60	70	80	90	100	110
c	0.7010	3.9976	9.0264	15.1764	22.0971	29.5716	37.4587

5. CONCLUDING REMARKS

In this paper, we have established a model, for the first time in the literature to our best knowledge, for the stock loan instrument. By relating the model to an American perpetual call option with a time-varying striking price, we have been able to derive explicitly the value of the loan, as well as the fair values of other key parameters.

There are many interesting research problems associated with a stock loan. For example, how to evaluate such a loan if any dividend income generated is credited toward accrued interest on the loan (rather than taken by the lender as in this paper)? How to model a termed loan and the associated decision problem where the client can choose to refinance and take out a larger loan for a subsequent term upon expiry of the current term? What if the lender may also terminate the contract at any time by paying a penalty to the borrower (in this case, the loan bears resemblance to the so-called *game options* as studied by Kifer 2000)?

To conclude, this paper is intended to be more initiating and inspiring—in the sense that it will inspire more researches along the line—than concluding and exhaustive.

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