Common Value Auctions with Return Policies

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Abstract

This paper examines the role of return policies in common value auctions. We first characterize the unique symmetric equilibrium in first-price and second-price auctions with continuous signals and discrete common values when certain return policies are provided. We then examine how the return policies affect a seller's revenue. When the lowest common value is zero, a more generous return policy generates a higher seller's revenue; the full refund policy extracts all the surplus and therefore implements the optimal selling mechanism; given any return policy, a second-price auction generates a higher revenue than a first-price auction. In a second-price auction where the lowest common value is not zero but still smaller than the seller's reservation value, a more generous return policy also generates a higher revenue. If the lowest common value is larger than the seller's reservation value, however, the optimal return policy could be a full refund, no refund or partial refund policy.

Key words: auctions, return policies, refund

JEL classification: D44, D72, D82

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1 Introduction

Traditional auctions have a history of thousands of years. Recently, the rapid growth of internet commerce makes online auctions extremely popular. These auctions create a problem for both the buyers and the sellers. As a buyer is unable to personally examine the good before bidding, he may find the good not exactly what he expects when he receives it. Even though this could also happen in brick-and-mortar store purchases, it is inarguably a more common problem in online auctions. Should the buyers be allowed to return the goods? Should the sellers keep some of the payments when the goods are returned? How would a return policy affect buyers' behavior? Would the sellers (and the buyers) benefit from such a return policy? These are some of the issues we will investigate in this paper.

Each day, there are millions of objects being auctioned on the internet through many online auction sites. A casual survey on eBay.com and Amazon.com shows that about half of the sellers provide a refund policy for returns, and the other half do not. On other online auction sites, we frequently find sellers providing very generous refund policies. For example, the National Hockey League online auctions provide a refund policy of 7-Day, 100% Money-Back Guarantee.

In a private-value auction, return policies do not affect a buyer's bidding strategy, since he never bids more than his valuation.[1] In contrast, with interdependent or common values, return policies induce bidders to bid more aggressively. Returns could happen with positive probability after the winning bidder receives the good and learns more information about its true value. In this paper, we focus on the common-value model in Wilson [12], since it is the simplest model accounting for interdependent and correlated values.[2] This model is widely used to model oil, gas and mineral rights auctions.

The phenomenon, known as the winner's curse, is well recognized in the auction literature. Winning could mean that the winner has overestimated the object value, since his bid is higher than those from other bidders. As the number of bidders increases, the winner's curse becomes more severe and bidders bid even more cautiously. However, if a return policy is in place, buyers will bid more aggressively, since the winner can get a refund by returning the object. A return policy acts as an insurance against overestimation and overcomes some of the winner's curse. In fact, a return policy can do more than mitigating the winner's curse. When the return policy is generous enough, for example, bidders may bid more than the unconditional estimates of the object value. If the seller implements the full refund policy, then it is obvious that bidders will bid very high in the auction. Of course, returns could negatively impact the seller's revenue as well as the efficiency of trading, as the seller usually

[1] Zhang [14] considers private values which are subject to idiosyncratic shocks after transaction, and illustrates how return policies can be part of the optimal mechanism.

[2] Resale can introduce common value components to a good of private value in nature. (See Haile [2], for example.) Milgrom and Weber [10] has a very general model of correlated values, but with return policies, it is difficult to characterize the equilibria. Nevertheless, the qualitative results should remain valid.

has a lower value for keeping the object. By selecting a proper return policy, the seller can achieve a higher revenue by balancing the trade off between higher bids and efficiency losses.

In general, a more generous return policy has three effects. First, it induces buyers to bid more aggressively. This is a positive effect. Second, the seller keeps a fraction of the transaction price when the winning bidder returns the object. This is a negative effect. We call the sum of these two effects the payment effect. It can be shown that the payment effect is usually positive. Third, returns change the efficiency of the object allocation. This is called the efficiency effect, which could either be positive or negative. How a return policy affects the seller's revenue depends on the magnitude of the payment effect and the efficiency effect.

In this paper, buyers receive independent signals conditional upon the true value of the object in our common-value auction models with return policies. To make the analysis as simple as possible, this common value is assumed to take discrete values, even though the signals are continuously distributed. We first consider the behavior of the bidders in a second-price auction, and then in a first-price auction. We compare the revenues generated by the two auction formats and find that a second-price auction generates more revenue than a first-price auction. In each auction format, as the seller promises more refund for returns, the seller's revenue is actually increased. Under certain conditions, the full refund policy extracts all the surplus from the bidders and implements the optimal selling mechanism.

There is a huge literature on auctions. However, none of the papers consider return policies. Huang, Qiu, and Matsubara [3] consider an algorithm for multi-unit auctions with partial refund for bid withdrawals that are caused by exogenous reasons. The paper provides an analysis from the perspectives of artificial intelligence, and bidders' strategic behaviors are not the focus. Zhang [14] and this paper analyze return policies in auctions by examining the strategic interactions among the bidders.

In theory, there exist optimal mechanisms for sellers to maximize revenue.[3] However, those optimal mechanisms are not commonly observed in reality, partly because too much detail regarding the underlining environment is required for the seller to design an optimal mechanism. The discrepancy between theory and common practice prompts the claim that a set of simplicity and robustness criteria should be imposed on the trading mechanisms.[4] Our auction model with return policies satisfies those simplicity criteria, and the return policies do not depend on much of the detail of the environment. As we shall show in this paper, return policies, while being \simple" instruments, are effective in revenue improving.

[3] The optimal auction with independent values has been established by Myerson [11]. Matthews [8] and Maskin and Riley [7] characterize the optimal mechanism with risk averse buyers and independent values. With correlated values, (almost) full surplus extraction can be achieved using the mechanism in Cremer and Mclean [1] and McAfee and Reny [9].

[4] Hurwicz [4] illustrates the need for mechanisms that are independent of the parameters of the model. Wilson [13] points out that a desirable property of a trading rule is that it “does not rely on features of the agents”. Lopomo [5] [6] restricts to mechanisms with “simplicity” and “robustness”.

The rest of this paper is organized as follows. In Section 2, we set up the model. In Section 3, we characterize the bidders' equilibrium strategies in second-price auctions. In Section 4, we characterize the bidders' equilibrium strategies in rst-price auctions. In Section 5, we establish the revenue ranking among di erent auction formats. In Section 6, we conclude. All proofs are relegated to an appendix.

2 The model

Suppose that there are two bidders, bidders 1 and 2. The common value of the object, V, can be either V_H or V_L, with $V_H > V_L$. Assume that $V = V_H$ with probability $_H$, and $V = V_L$ with probability $_L = 1 - {}_H$. Bidders know the distribution of the common value but not its true value before the bidding starts. Bidder i receives a signal x_i, $i = 1; 2$. This signal is correlated to the common value, V, but independently distributed across bidders conditional on V. If $V = V_H$, then x_i follows the distribution with p.d.f. $f_H(\cdot)$ and c.d.f. $F_H(\cdot)$. If $V = V_L$, then x_i follows the distribution with p.d.f. $f_L(\cdot)$ and c.d.f. $F_L(\cdot)$. Assume that $F_H(\cdot)$ and $F_L(\cdot)$ have a common support, $[\underline{x}; \overline{x}]$. To ensure that a higher signal implies a higher probability of $V = V_H$, we assume that $(x) = \frac{f_H(x)}{f_L(x)}$ is increasing in x, i.e., F_H dominates F_L in likelihood ratio. The lemma below lists a few properties implied by this assumption. The proof is standard and is thus omitted.

Lemma 1 *Suppose that $F_H(x)$ dominates $F_L(x)$ in likelihood ratio, i.e., (x) is increasing in x. Then*

1. *F_H dominates F_L in hazard rate, i.e. $\frac{f_H}{1 \; F_H} \leq \frac{f_L}{1 \; F_H}$.*
2. *F_H dominates F_L in reversed hazard rate, i.e. $\frac{f_H}{F_H} \geq \frac{f_L}{F_L}$.*
3. *$\frac{F_H}{F_L}$ is increasing.*

Now we consider the return policy. Let p be the transaction price in the auction. Suppose that the seller charges a fee $c = \; p$ if the winning bidder (winner) returns the object, where $\in [0; 1]$. Then the winner receives a refund of $(1 - \;)p$. Here, we assume that there is no transaction cost for returns on both sides. We focus on the case where > 0. If $= 0$, there could be multiple equilibria with every bidder bidding greater than or equal to V_H.[5]

The timing of the game is as follows.

1. Nature moves rst and selects $V = V_H$ or $V = V_L$. Conditional on V, each bidder draws a signal independently.

[5] As we shall show later, when converges to zero, the equilibrium converges to the one where every bidder bids V_H. Furthermore, bidding more than V_H is weakly dominated by bidding V_H. Therefore, when $= 0$, we pick the undominated equilibrium where every bidder bids V_H.

2. Either a first-price or a second-price auction with return policy   is held, and the winner is determined.

3. The winner learns the true value of the object and decides whether or not to return the object for a refund.

We assume that the winner can learn about the true value V after the auction ends. This assumption is motivated by the fact that in online auctions, after a buyer receives the object, he would learn more about its value. In auctions for oil, gas and mineral rights, the winners will learn more information by doing more testing and uncertainties resolve over time.

In the following analysis, we will focus on the symmetric perfect Bayesian equilibrium with strictly increasing bidding function in the auction. We will start by analyzing the last stage of the game, where the winner makes the return decision. In the following section, we will first examine the second-price auctions. We will then examine the first-price auctions in the subsequent section.

3 Second-price auctions

In a second-price auction, the transaction price is equal to the second highest bid in the auction. We assume that both bidders adopt the same strictly increasing bidding function $B^S(\cdot)$ in the auction stage. We can restrict our attention to bidding functions taking values in $[V_L; V_H]$. This is because a buyer with the highest signal should not bid more than V_H; bidding more than V_H would sometimes gives him a negative surplus and is dominated by bidding V_H; if a bidder with the lowest signal bids less than V_L, then by increasing his bid to V_L he may win with a positive probability and thus get a positive surplus.

Now consider the return stage. Assume that buyer 1 is the winner and his signal is x. Suppose that he bids $B^S(x)$, wins the auction, and pays $B^S(x_2)$. If the realization of the value of the object is $V = V_H$, he will not return it since his payment is less than V_H. If $V = V_L$, he returns the object when $V_L < (1-\ )B^S(x_2)$, i.e., $x_2 > \left(B^S\right)^{-1}(\frac{V_L}{1-\ })$; otherwise, he keeps the object. As a result, there can only be three different situations in the return stage. Case 1: $\left(B^S\right)^{-1}(\frac{V_L}{1-\ }) \geq \overline{x}$, and thus the winner keeps the object all the time. Case 2: $\left(B^S\right)^{-1}(\frac{V_L}{1-\ }) \leq \underline{x}$, and thus the winner returns the object whenever $V = V_L$. Case 3: $\underline{x} < \left(B^S\right)^{-1}(\frac{V_L}{1-\ }) < \overline{x}$, and the winner's return decision is based on a cutoff rule when $V = V_L$.

Note that in a two-bidder second-price common-value auction without return policies, from Milgrom and Weber [10], a bidder with signal x would bid the expected object value conditional on the other bidder having the same signal x:

$$E(V|x;x) = \frac{{}_H V_H f_H(x)^2 + \;\; {}_L V_L f_L(x)^2}{{}_H f_H(x)^2 + \;\; {}_L f_L(x)^2} = \frac{{}_H V_H \;\; (x)^2 + \;\; {}_L V_L}{{}_H \;\; (x)^2 + \;\; {}_L}: \quad (1)$$

De ne

$$^S(x) \equiv 1 - \frac{V_L}{E(V|x;x)} = \frac{(V_H - V_L) \;\; {}_H \;\; (x)^2}{V_H \;\; {}_H \;\; (x)^2 + \;\; {}_L V_L} \quad (2)$$

as the winning bidder's loss as a percentage to the bidder's bid when the value of the object turns out to be V_L. There are two cuto s for that are important in the characterization of the bidders' equilibrium bidding function. (Recall that is the percentage of the transaction price that is retained by the seller if the winning bidder returns the object.)

$$\underline{\;\;}^S = \;\; {}^S(\underline{x}); \quad (3)$$

$$\overline{\;\;}^S = \;\; {}^S(\overline{x}): \quad (4)$$

The type of equilibrium we will obtain depends crucially on the value of . As shown in the proof of Proposition 1, it turns out that if is lower than $\underline{\;\;}^S$, the winning bidder would always return the object when $V = V_L$. If is higher than $\overline{\;\;}^S$, the winning bidder would never return the object when $V = V_L$. If is intermediate, the winning bidder would sometimes return the object when $V = V_L$. Furthermore, the intervals for in the above three cases do not overlap with each other and they cover the entire interval of $(0;1]$. Thus we can conclude that a unique symmetric perfect Bayesian Nash equilibrium exists for any $\in (0;1]$. De ne x^S as the solution to $= \;\; {}^S(x^S)$ for $\in (\underline{\;\;}^S; \overline{\;\;}^S)$. Since $^S(x)$ is strictly increasing, x^S is unique and belongs to $(\underline{x}; \overline{x})$. These results are characterized in the following proposition.

Proposition 1 *In the second-price common value auction with return policy , the unique symmetric equilibrium is characterized as follows in three cases.*

Case 1: If $\geq \overline{\;\;}^S$, *each bidder adopts the following strictly increasing bidding function:*

$$B^S(x) = B^{S1}(x) \equiv E(V|x;x) = \frac{{}_H V_H \;\; (x)^2 + \;\; {}_L V_L}{{}_H \;\; (x)^2 + \;\; {}_L}: \quad (5)$$

The winning bidder never returns the object.

Case 2: If $\leq \underline{\;\;}^S$, *each bidder adopts the following strictly increasing bidding function:*

$$B^S(x) = B^{S2}(x) = \frac{{}_H V_H \;\; (x)^2}{{}_H \;\; (x)^2 + \;\; {}_L}: \quad (6)$$

The winning bidder always returns the object whenever $V = V_L$.

Case 3: If $\underline{\ }^S < \quad < \overline{\ }^S$, each bidder adopts the following strictly increasing bidding function:

$$B^S(x) = \begin{cases} B^{S1}(x); & \text{if } x \leq x^S \ ; \\ B^{S2}(x); & \text{if } x \geq x^S \ : \end{cases} \tag{7}$$

The winning bidder returns the object when $V = V_L$ if he pays more than $B^S(x^S\ )$.

In Case 1 of the above proposition, since the return policy is never executed, the bidding function coincides with the one with no return policy. Obviously, providing a very strict return policy is equivalent to no returns. The bidding function $B^{S1}(x)$ corresponds to the one in Milgrom and Weber [10].

In Case 2, the function $B^{S2}(x)$ is in fact the equilibrium bidding function for the game when the winner is forced to return the object if the realized value is V_L. It is equal to the price that the bidder will break even if he pays that price (i.e., the other bidder also has signal x) and gets V_H when $V = V_H$, and pays percent of that price and gets 0 when $V = V_L$. That is, $B^{S2}(x)$ is the solution to $(V_H - B) \ _H f_H(x)^2 + (- \ B) \ _L f_L(x)^2 = 0$. When $V_L = 0$, this bid is equivalent to the expected object value of a bidder with signal x conditional on the other bidder having the same signal x and returning the object with probability $1 -$ when $V = V_L$.

In Case 3, whether the winning bidder returns the object or not depends on how much he pays. It is easy to show that $B^{S1}(x) \geq B^{S2}(x)$ for $x \leq x^S$ and $B^{S1}(x) \leq B^{S2}(x)$ for $x \geq x^S$ with equality at the cuto x^S . Therefore, the bidding function is the maximum of the two functions in Cases 1 and 2. However, as we shall show in the next section, this pattern is not valid for rst-price auctions.

Given any return policy $\in (0; 1]$, there exists a unique symmetric equilibrium. When $\to 0$, $B^S(x) \to V_H$. We know that when $= 0$, there are many equilibria. First, bidding V_H and the winner always returns the object when $V = V_L$ and keeps the object when $V = V_H$ is an equilibrium. Second, bidding any amount more than V_H and the winner returns the object

4 First-price auctions

Now we examine rst-price auction with returns. Here, the transaction price is the winning bid. Again, we focus on an equilibrium where every bidder adopts the same strictly increasing bidding function $B^F(\cdot)$. Similarly to the second-price auctions, we can establish the range of the bidding function to be a subset of $[V_L; V_H]$.

We rst examine the winning bidder's return decision. Suppose that a bidder has signal x but bids $B^F(\tilde{x})$. If he wins, he pays $B^F(\tilde{x})$ for the object. When the realization of the object value is $V = V_H$, he will keep the object, since he pays less than V_H. When $V = V_L$, he will return the object if $V_L < (1 - \ )B^F(\tilde{x})$, i.e., $\tilde{x} > (B^F)\ ^1(\frac{V_L}{1\ })$. As a result, there can only be three di erent situations in the return stage. Case 1: $(B^F)\ ^1(\frac{V_L}{1\ }) \geq \bar{x}$, and thus the winner keeps the object all the time. Case 2: $(B^F)\ ^1(\frac{V_L}{1\ }) \leq \underline{x}$, and thus the winner returns the object all the time when $V = V_L$. Case 3: $\underline{x} < (B^F)\ ^1(\frac{V_L}{1\ }) < \bar{x}$, and thus the winner's return decision is a cuto rule when $V = V_L$.

De ne two functions:

$$L_1(\ |x) = e^{\ \int^x \frac{_H f_H(s)^2 + \ _L f_L(s)^2}{_H f_H(s) F_H(s) + \ _L f_L(s) F_L(s)} ds}; \tag{8}$$

$$L_2(\ |x;\ ) = e^{\ \int^x \frac{_H f_H(s)^2 + \ _L f_L(s)^2}{_H f_H(s) F_H(s) + \ _L f_L(s) F_L(s)} ds} \tag{9}$$

Lemma 2 $L_1(\ |x)$ *and* $L_2(\ |x;\ )$ *are both proper c.d.f.'s of with support* $[\underline{x}; x]$.

Note that in a two-bidder rst-price common-value auction without any return policies, according to Milgrom and Weber [10], all bidders would bid according to the same strictly increasing function:

$$\int_{\underline{x}}^{x} E(V|\ ;\ )dL_1(\ |x) = \int_{\underline{x}}^{x} \frac{_H V_H\ (\ )^2 + \ _L V_L}{_H\ (\ )^2 + \ _L} dL_1(\ |x):$$

De ne

$$^F(x) \equiv 1 - \frac{V_L}{\int_{\underline{x}}^{x} E(V|\ ;\ )dL_1(\ |x)} = \frac{\int_{\underline{x}}^{x} \frac{(V_H\ V_L)\ _H\ (\ )^2}{V_H\ _H\ (\ )^2 + \ _L} dL_1(\ |x)}{\int_{\underline{x}}^{x} \frac{_H V_H\ (\ )^2 + \ _L V_L}{_H\ (\ )^2 + \ _L} dL_1(\ |x)}$$

to be the winning bidder's loss as a percentage to the bidder's bid when the value of the object turns out to be V_L. Since the bidding function (??) is strictly increasing, $^F(x)$ is strictly increasing. As in the second-price auctions, there are two cuto s for that are important in the characterization of the bidders' equilibrium bidding function.

$$\underline{\ }^F = \ ^F(\underline{x});$$

$$\overline{}^{F} = \ ^{F}(\overline{x}):$$

Note that $^{F}(\underline{x}) = \ ^{S}(\underline{x})$.

As in the second-price auctions, plays an important role in the return decision. De ne x^{F} as the solution to $= \ ^{F}(x^{F}\ )$ for $\in (\underline{}^{F}; \overline{}^{F})$. Since $^{F}(x)$ is strictly increasing, x^{F} is unique and belongs to $(\underline{x}; \overline{x})$. Also de ne

$$A = \frac{V_L}{1 -} - \int_{\underline{x}}^{x^{F*}} \frac{{}_H V_H\ (\ )^2}{{}_H\ (\ )^2 + \ {}_L} dL_2(\ |x^{F}\ ;\ ):$$

Similarly to the second-price auctions, we have the following proposition.

Proposition 2 *In a rst-price common value auction with return policy , the unique symmetric equilibrium can be characterized as follows in three cases.*
Case 1: When $\geq \overline{}^{F}$; each bidder adopts the following strictly increasing bidding function:

$$B^{F}(x) = B^{F1}(x) = \int_{\underline{x}}^{x} \frac{{}_H V_H\ (\ )^2 + \ {}_L V_L}{{}_H\ (\ )^2 + \ {}_L} dL_1(\ |x): \tag{10}$$

The winning bidder never returns the object.

Case 2: When $\leq \underline{}^{F}$; each bidder adopts the following strictly increasing bidding function:

$$B^{F}(x) = B^{F2}(x) = \int_{\underline{x}}^{x} \quad V_H$$

For the same reason as in the second-price auctions, when = 0, we choose the equilibrium with $B^F(x) = V_H$ and the winner always returns the object whenever $V = V_L$ and keeps the object whenever $V = V_H$ as the equilibrium. This is captured by Case 2 with = 0 in the above proposition.

5 Revenue ranking

In this section, we shall make several comparisons in revenue. Suppose that the seller's reservation value of the object is V_0, where $V_0 < V_H$. Consider a return policy with percentage fee of for returns. Suppose that the seller implements a more generous return policy (i.e., a lower). There are three effects. First, buyers bid more aggressively. This is a positive effect. Second, when the winner returns the object, the seller keeps only a smaller fraction of the transaction price. This is a negative effect. Third, because the cost for returning is lower, the probability of the object being returned to the seller is higher. The effect on the efficiency of the object allocation could be positive or negative, depending on whether the seller's reservation value is higher or lower than the object value. We call the sum of the first two effects the payment effect, and the third effect the efficiency effect. The seller can improve her revenue by balancing the trade-off between the payment effect and the efficiency effect.

In what follows, we shall first examine the special case of $V_L = 0$, and then the general case. We denote the case of $V_L = 0$ the benchmark. In this case, the object is either in perfect condition (high common value), or totally useless (zero common value). The winner always keeps the object when the realized common value is high, and always returns it when the realized common value is zero, regardless of the return policy. Because the return policy does not alter the allocation of the object, there is no efficiency variation among different return policies. This allows us to focus on the role of return policy on the payment effect. Analytically, we can make use of the linkage principle and show how the return policy affects the revenue in this case.[6]

5.1 The linkage principle in the benchmark

Consider a direct mechanism of our model. Let $M(\tilde{x}; x)$ be the expected payment by a bidder with signal x but reported $\tilde{x}$. We have the following proposition.

Proposition 3 *Let A and B be two auctions with return policies. In both auctions, the bidder with the highest bid wins. Furthermore, the winner always keeps the object if $V = V_H$, and always returns it if $V = V_L$. Suppose that in each auction, there is a symmetric*

[6]Unfortunately, when $V_L \neq 0$, the allocation of the object is affected by the return policy and the linkage principle does not apply.

and strictly increasing equilibrium bidding function with the properties that (i) for all x, $M_2^A(x;x) \geq M_2^B(x;x)$*; (ii)* $M^A(\underline{x};\underline{x}) = M^B(\underline{x};\underline{x}) = 0$*. Then the seller's expected revenue from A is at least as large as the expected revenue from B.*

In the benchmark, $V_L = 0$. Given any return policy and auction format, the winner with the highest bid wins; and the winner always returns the object when $V = V_L$ and always keeps the object when $V = V_H$. Thus, both rst-price and second-price auctions with any return policy can be regarded as a mechanism in the above proposition. Note that the equilibrium strategy for a second-price auction is characterized by Case 2 in Proposition 1, and the equilibrium strategy for a rst-price auction is characterized by Case 2 in Proposition 2.

We can rank the expected revenues in the second-price auctions with di erent s as follows.

Proposition 4 *Suppose that* $V_L = 0$*. Then the seller's revenue is decreasing in in the second-price auctions. That is, the more generous the return policy is, the more revenue a second-price auction generates.*

The intuition behind this proposition is as follows. Bidders take into consideration of the possible $V_L = 0$ when they calculate their bids. The only situation that a ects the seller's revenue is when there is some probability that the winning bidder will return the object. Denote a bidder's bid in this case as $B^{S2}(x; \ )$. It is the solution to

$$\left[V_H - B^{S2}(x; \ )\right] \frac{{}_H f_H(x)^2}{{}_H f_H(x) + \ {}_L f_L(x)} + \left[- \ B^{S2}(x; \ )\right] \frac{{}_L f_L(x)^2}{{}_H f_H(x) + \ {}_L f_L(x)} = 0;$$

or equivalently,

$$f_L(x)\left[B^{S2}(x; \ ) \ {}_H f_H(x) \ (x) + \ B^{S2}(x; \ ) \ {}_L f_L(x)\right] = V_H \ {}_H f_H(x)^2; \qquad (13)$$

where $(x) = \frac{f_H(x)}{f_L(x)}$.

When decreases, $B^{S2}(x; \ )$ needs to increase to keep the equation binding. Furthermore, because $B^{S2}(x; \ )$ has increased, $B^{S2}(x; \ )$ needs to decrease to keep the equation binding. Now consider the seller's revenue. The seller receives either $B^{S2}(x; \ )$ when $V = V_H$ or $B^{S2}(x; \ )$ when $V = V_L = 0$, if the other bidder's signal (denoted by y) is higher. In this case, the other bidder wins and pays $B^{S2}(x; \ )$. The relevant expression entering the seller's expected revenue becomes

$$\begin{aligned} & B^{S2}(x; \ )\frac{{}_H f_H(y) f_H(x)}{{}_H f_H(x) + \ {}_L f_L(x)} + \ B^{S2}(x; \ )\frac{{}_L f_L(y) f_L(x)}{{}_H f_H(x) + \ {}_L f_L(x)} \\ = & \frac{f_L(y)}{{}_H f_H(x) + \ {}_L f_L(x)}\left[B^{S2}(x; \ ) \ {}_H f_H(x) \ (y) + \ B^{S2}(x; \ ) \ {}_L f_L(x)\right]; \end{aligned} \qquad (14)$$

where $y > x$. Note our previous assumption that (x) is an increasing function, implying $(y) > (x)$. Therefore, when decreases, the increase in B^{S2} together with the decrease in B^{S2} keeping the left-hand side of (13) constant will increase the value of (14), and thus will increase the seller's revenue. Intuitively, because the seller receives the bid of a first bidder only when the second bidder has a higher signal, and this higher signal makes $V = V_H$ more likely to occur than the probability used in the first bidder's calculation, the total expected revenue for the seller is higher.

Similar ranking in the first-price auctions with different s can be obtained as well. We have the following proposition.

Proposition 5 *Suppose that $V_L = 0$. Then the seller's revenue is decreasing in in the first-price auctions. That is, the more generous the return policy is, the more revenue a first-price auction generates.*

The above two propositions illustrate that the payment effect is positive in both auctions when $V_L = 0$. Since the efficiency effect vanishes when $V_L = 0$, a more generous return policy increases the seller's revenue. This implies that the full refund policy ($= 0$) is the best return policy. With the full refund policy, in both auctions, all buyers bid up to V_H; and the winner keeps the object when $V = V_H$ and returns it when $V = V_L = 0$. Thus, the seller extracts all possible surplus and the buyers earn zero surplus. As a result, both auctions with full refund policy implement the optimal mechanism which extracts all surplus. This is summarized in the following corollary.

Corollary 1 *Suppose that $V_L = 0$. Then either the first-price or second-price auction with full refund policy implements the optimal mechanism.*

In what follows, we compare the revenues in the first-price and the second-price auctions given the same . We have the following proposition.

Proposition 6 *Suppose that $V_L = 0$. Given the same return policy , a second-price auction generates at least as much revenue as a first-price auction.*

This proposition shows that the result in Milgrom and Weber [10] that second-price auctions generates weakly more revenue than first-price auctions can be generalized to auctions with return policies. In Milgrom and Weber [10], the result can be derived directly from the linkage principle. In contrast, the linkage principle cannot be applied to auctions with return policies. This is because for the linkage principle to work, the difference between the two expected payment functions must be increasing in a bidder's reported type. This property is satisfied among second-price auctions with different s, as well as among first-price auctions with different s. But when we compare a first-price auction with a second-price auction with the same , the property is no longer valid. (See the proof of this proposition for details.) Nevertheless, revenue ranking is still possible here. This is because in a second-price auction, the seller receives the bid of a bidder only when the other bidder has a higher signal, and this higher signal makes $V = V_H$ more likely to happen than the first bidder originally thought. That is, the object gets returned less often in the seller's revenue calculation than in a bidder's surplus calculation. However, this effect is absent in the first-price auction. Therefore, the total expected revenue for the seller is higher in the second-price auction.

5.2 The general case

Now we consider the general case, where $V_L \neq 0$. In this case, the linkage principle does not apply, since different return policies generate different allocations of the object being auctioned. As the equilibrium bidding function in first-price auctions is very complex, we focus on the second-price auctions here. First-price auctions should have qualitatively similar results. We have the following proposition which is similar to Proposition 4.

Proposition 7 *Suppose that $V_L \leq V_0$. Then the seller's revenue is decreasing in in the second-price auctions. That is, the more generous the return policy is, the more revenue the auction will generate. Furthermore, the second-price auction with the full refund policy implements the optimal mechanism.*

As is shown in the proof, the payment effect is always positive in the second-price auctions. When $V_L \neq 0$ but $V_L \leq V_0$, the efficiency effect is positive since the return policy improves efficiency. Therefore, the total effect is positive, which implies that a more generous return policy again increases the seller's revenue and the full refund policy is optimal. The intuition for this proposition is exactly the same as Proposition 4.

When $V_L > V_0$, the efficiency effect is negative. In this case, the net effect of a more generous return policy depends on which of the two effects (the payment effect and the efficiency effect) dominates. The following example shows that the optimal return policy can be a full refund, no refund or partial refund policy.

Example: Suppose that $V_0 = 0$, $V_H = 1$, V_L to be specified, with ${}_H = {}_L = 0.5$. For $x \in [0, 1]$, $F_H(x) = x^2$, $F_L(x) = 2x - x^2$, Then $f_H(x) = 2x$, $f_L(x) = 2 - 2x$, and $(x) = \frac{x}{1-x}$. Note that (x) is indeed strictly increasing as we previously assumed. We will vary the value of V_L and let V_L take the values of 0.02, 0.25, 0.28, and 0.5, respectively. Figure 1 puts all values of V_L in one figure, while the rest of the figures each illustrate one value of V_L. When $V_L = 0.02$, the seller's revenue is decreasing and then increasing in with the minimum reached at $= 0.97$; the optimal return policy is the full refund policy. When $V_L = 0.25$, the seller's revenue first increases, then decreases, and then increases in ; the optimal return policy is a partial refund policy with $= 0.82\%$. When $V_L = 0.28$, the seller's revenue first increases, then decreases, and then increases in ; the optimal return policy is the no refund policy. When V_L = 0:5, the seller's revenue is increasing in ; the no refund policy is optimal again.

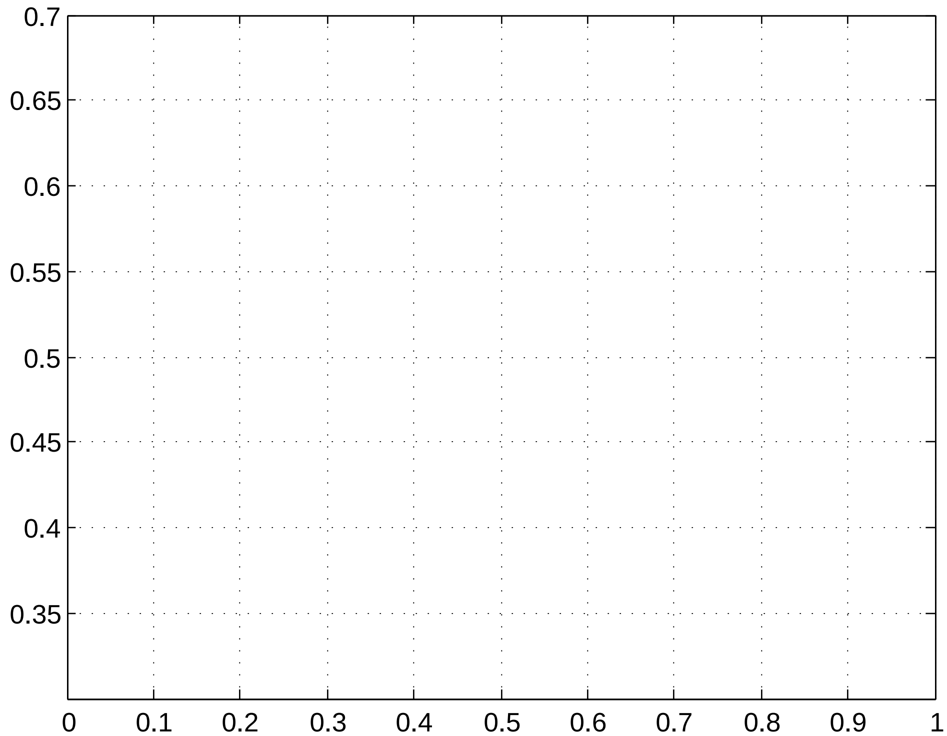

Figure 1: The seller's revenue as function of for di erent V_L

6 Conclusion

This paper investigates how return policies a ect buyers' bidding strategies in rst-price and second-price auctions and the respective seller's revenue. Providing a return policy undoubtedly induces buyers to bid more aggressively. When the lowest value of the object is zero, the more generous a return policy is, the more expected revenue the auction generates. This is true for both the rst-price and the second-price auctions. The standard results in Milgrom and Weber [10] that second-price auction generate more revenue than rst-price auctions in common value auctions can be extended to the case of return policies. When the lowest value of the object is non-zero but still lower than the seller's reservation value, the revenue is again higher when the return policy is more generous in the second-price auctions.

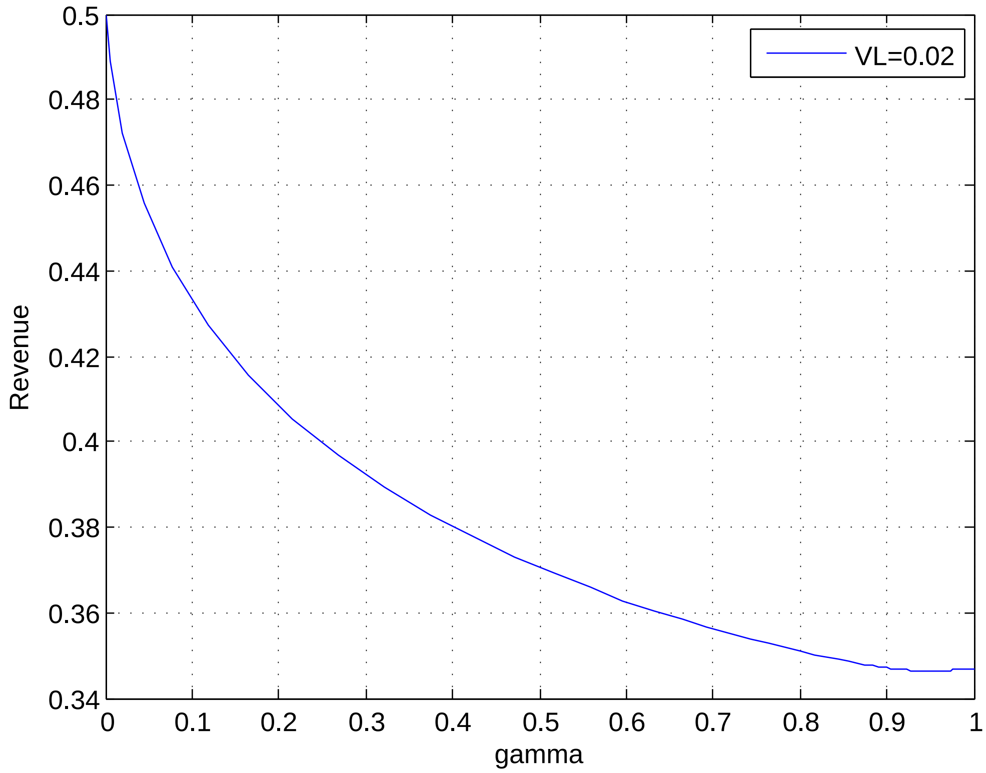

Figure 2: The seller's revenue as function of γ when $V_L = 0.02$

Auctions with return policies are more complicated to analyze than standard auctions, as the winning bidder may return the object when he obtains more information regarding the object value. Therefore, a higher bid induced by a more generous return policy may not be beneficial to the seller. This paper shows that when the efficiency losses from the returns are not significant, a more generous return policy helps the seller. Since a seller can also use return policies to signal the quality of the object, we should expect to see return policies in many auctions as we have witnessed in online auctions, where buyers have less confidence in the quality of the objects.

7 Appendix

Proof for Proposition 1

Case 1: Never return

We first characterize the symmetric equilibrium bidding function in the case where the winning bidder never returns the object after winning. Let $B^{S1}(\cdot)$ denote the bidding function

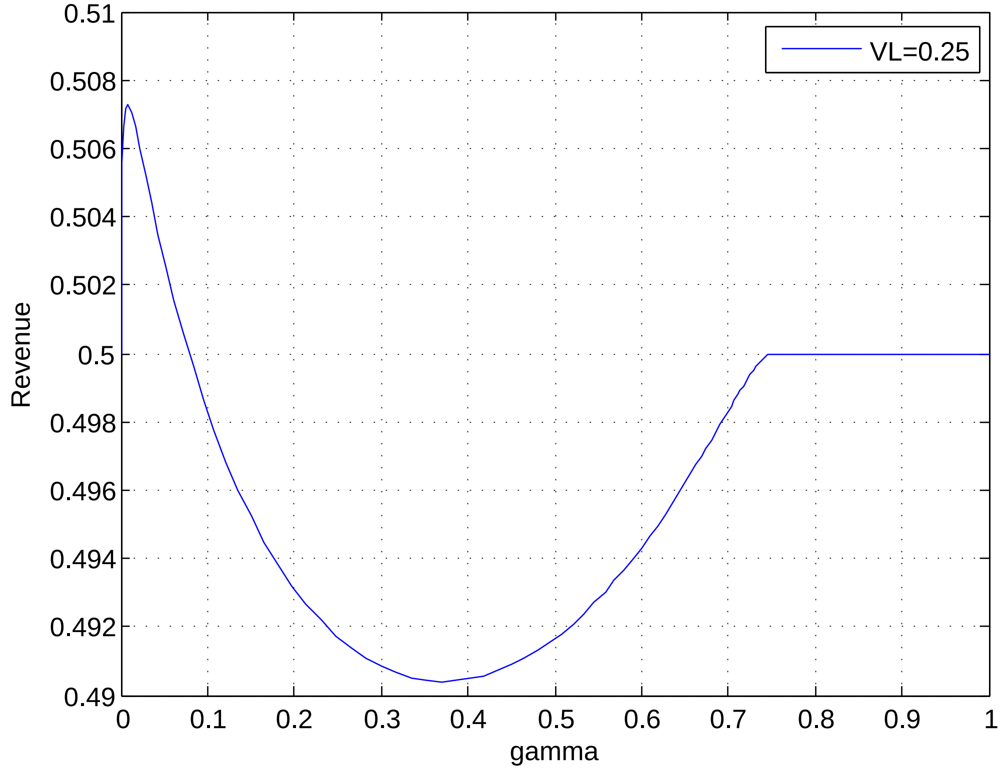

Figure 3: The seller's revenue as function of   when $V_L = 0.25$

in this case. Consider buyer 1. Suppose that buyer 1's signal is x and he pretends to have signal $\tilde{x}$ and bids $B^{S1}(\tilde{x})$. Given that when the realization of the value is V_L, bidder 1 will keep the object if he wins, his expected surplus in the auction is given by:

$$
\begin{aligned}
{}^{S1}(x;\tilde{x}) &= Pr(V = V_H|x_1 = x)E\{[V - B^{S1}(x_2)]I\{x_2 < \tilde{x}\}|x_1 = x; V = V_H\} && (15)\\
&\quad + Pr(V = V_L|x_1 = x)E\{[V - B^{S1}(x_2)]I\{x_2 < \tilde{x}\}|x_1 = x; V = V_L\} && (16)\\
&= {}_H(x)\int_{\underline{x}}^{\tilde{x}}[V_H - B^{S1}(x_2)]dF_H(x_2) + {}_L(x)\int_{\underline{x}}^{\tilde{x}}[V_L - B^{S1}(x_2)]dF_L(x_2); && (17)
\end{aligned}
$$

where

$$
\begin{aligned}
{}_H(x) &\equiv \Pr(V = V_H|x_1 = x) && (18)\\
&= \frac{\Pr(x_1 = x|V = V_H)\Pr(V = V_H)}{\Pr(x_1 = x|V = V_H)\Pr(V = V_H) + \Pr(x_1 = x|V = V_L)\Pr(V = V_L)} && (19)\\
&= \frac{f_H(x)\ {}_H}{f_H(x)\ {}_H + f_L(x)\ {}_L}; && (20)
\end{aligned}
$$

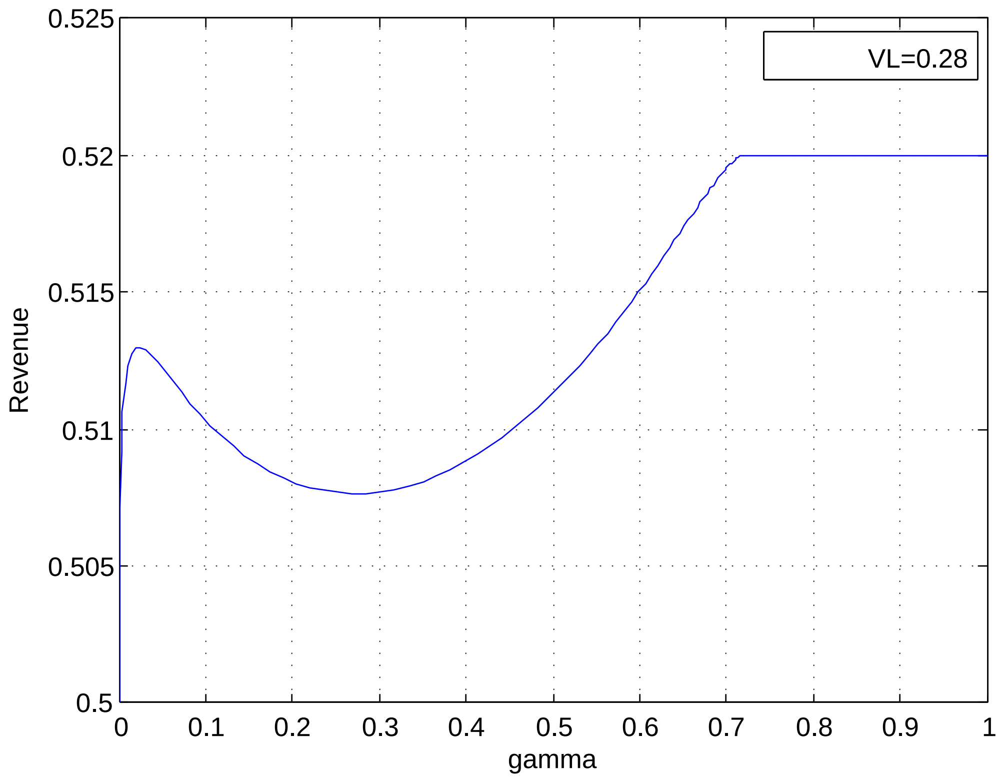

Figure 4: The seller's revenue as function of   when $V_L = 0{:}28$

and where ${}_L(x) = \Pr(V = V_L | x_1 = x) = 1 - {}_H(x)$. It is important to note that ${}_H(x)$ is increasing in x and ${}_L(x)$ is decreasing in x. Therefore,

$$
\begin{aligned}
& \frac{@ \;\; {}^{S1}(x; \not{x})}{@\not{x}} \\
= \;& {}_H(x)[V_H - B^{S1}(\not{x})]f_H(\not{x}) + {}_L(x)[V_L - B^{S1}(\not{x})]f_L(\not{x}) \\
= \;& [\, {}_H(x)f_H(\not{x}) + {}_L(x)f_L(\not{x})] \left[\frac{{}_H(x)V_H f_H(\not{x}) + {}_L(x)V_L f_L(\not{x})}{{}_H(x)f_H(\not{x}) + {}_L(x)f_L(\not{x})} - B^{S1}(\not{x}) \right] \\
= \;& [\, {}_H(x)f_H(\not{x}) + {}_L(x)f_L(\not{x})] \left[\frac{{}_H f_H(x)V_H f_H(\not{x}) + {}_L f_(x)V_L f_L(\not{x})}{{}_H f_H(x)f_H(\not{x}) + {}_L f_L(x)f_L(\not{x})} - B^{S1}(\not{x}) \right] \\
= \;& [\, {}_H(x)f_H(\not{x}) + {}_L(x)f_L(\not{x})] \left[\frac{{}_H V_H \;\; (x) \;\; (\not{x}) + {}_L V_L}{{}_H \;\; (x) \;\; (\not{x}) + {}_L} - B^{S1}(\not{x}) \right] : \qquad (21)
\end{aligned}
$$

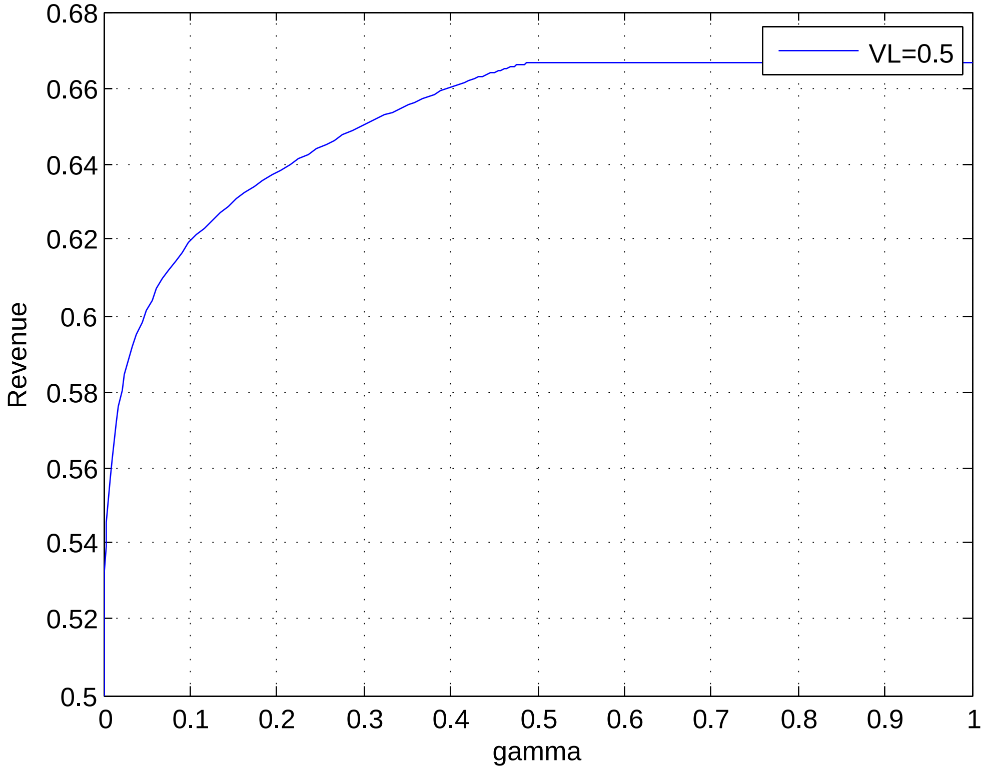

Figure 5: The seller's revenue as function of γ when $V_L = 0.5$

The first order condition (FOC) for this bidder's surplus maximization problem gives:

$$\left.\frac{\partial \pi^{S1}(x;\tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=x} = 0. \tag{22}$$

Solving for $B^{S1}(x)$, we have

$$B^{S1}(x) = \frac{\alpha_H V_H \phi(x)^2 + \alpha_L V_L}{\alpha_H \phi(x)^2 + \alpha_L}. \tag{23}$$

The FOC is usually only a necessary condition. We shall show below that the FOC is also a sufficient condition for the above maximization problem. It is easy to check that $\frac{\alpha_H V_H \phi(x)\phi(\tilde{x})+\alpha_L V_L}{\alpha_H \phi(x)\phi(\tilde{x})+\alpha_L}$ is increasing in x. Therefore, given the bidding function defined in equation (23), the surplus function $\pi^{S1}(x;\tilde{x})$ is a unimodal function with the maximum at $\tilde{x} = x$; i.e., increasing for $\hat{x} \leq x$ and decreasing for $\hat{x} \geq x$. To see this, for $\hat{x} \leq x$,

$$
\begin{aligned}
& \frac{\partial \ ^{S1}(x;\hat{x})}{\partial \hat{x}} && (24)\\
= & \left[\ _H(x) f_H(\hat{x}) + \ _L(x) f_L(\hat{x})\right] \left[\frac{\ _H V_H \ (x)\ (\hat{x}) + \ _L V_L}{\ _H \ (x)\ (\hat{x}) + \ _L} - \frac{\ _H V_H \ (\hat{x})\ (\hat{x}) + \ _L V_L}{\ _H \ (\hat{x})\ (\hat{x}) + \ _L} \right] && (25)\\
\geq & \left[\ _H(x) f_H(\hat{x}) + \ _L(x) f_L(\hat{x})\right] \left[\frac{\ _H V_H \ (\hat{x})\ (\hat{x}) + \ _L V_L}{\ _H \ (\hat{x})\ (\hat{x}) + \ _L} - \frac{\ _H V_H \ (\hat{x})\ (\hat{x}) + \ _L V_L}{\ _H \ (\hat{x})\ (\hat{x}) + \ _L} \right] && (26)\\
= & \ 0; && (27)
\end{aligned}
$$

and for $\hat{x} \geq x$,

$$
\begin{aligned}
& \frac{\partial \ ^{S1}(x;\hat{x})}{\partial \hat{x}} && (28)\\
= & \left[\ _H(x) f_H(\hat{x}) + \ _L(x) f_L(\hat{x})\right] \left[\frac{\ _H V_H \ (x)\ (\hat{x}) + \ _L V_L}{\ _H \ (x)\ (\hat{x}) + \ _L} - \frac{\ _H V_H \ (\hat{x})\ (\hat{x}) + \ _L V_L}{\ _H \ (\hat{x})\ (\hat{x}) + \ _L} \right] && (29)\\
\leq & \left[\ _H(x) f_H(\hat{x}) + \ _L(x) f_L(\hat{x})\right] \left[\frac{\ _H V_H \ (\hat{x})\ (\hat{x}) + \ _L V_L}{\ _H \ (\hat{x})\ (\hat{x}) + \ _L} - \frac{\ _H V_H \ (\hat{x})\ (\hat{x}) + \ _L V_L}{\ _H \ (\hat{x})\ (\hat{x}) + \ _L} \right] && (30)\\
= & \ 0. && (31)
\end{aligned}
$$

Therefore, $\hat{x} = x$ is indeed optimal and the sufficiency of FOC for the maximization is confirmed. Of course, for the above bidding function to be an equilibrium, we need to guarantee that the winner never wants to return the object. Note that the bidding function is increasing. The condition of no return is equivalent to

$$
\begin{aligned}
& V_L \geq (1 - \ )B^{S1}(\bar{x}) && (32)\\
\Leftrightarrow \quad & \ \geq \frac{(V_H - V_L)\ _H \ (\bar{x})^2}{V_H \ _H \ (\bar{x})^2 + \ _L V_L} \equiv \bar{\ }^S. && (33)
\end{aligned}
$$

Case 2: Always return when $V = V_L$

In this case, the winning bidder always returns the object when $V = V_L$. Given this, buyer 1's surplus when he pretends to have signal $\hat{x}$ is given by

$$
\begin{aligned}
\ ^{S2}(x;\hat{x}) = \ & Pr(V = V_H | x_1 = x) E\{[V - B^{S2}(x_2)] I\{x_2 < \hat{x}\} | x_1 = x; V = V_H\} && (34)\\
& + Pr(V = V_L | x_1 = x) \Big[- E\{\ B^{S2}(x_2) I\{x_2 < \hat{x}\} | x_1 = x; V = V_L\} \Big] && (35)\\
= \ & \ _H(x) \int_{\underline{x}}^{\hat{x}} [V_H - B^{S2}(x_2)] dF_H(x_2) - \ _L(x) \int_{\underline{x}}^{\hat{x}} \ B^{S2}(x_2) dF_L(x_2). && (36)
\end{aligned}
$$

Taking the derivative with respect to $\hat{x}$, we have

$$
\frac{\partial \ ^{S2}(x;\hat{x})}{\partial \hat{x}}
$$

$$
\begin{aligned}
&= \quad {}_H(x)[V_H - B^{S2}(\tilde{x})]f_H(\tilde{x}) - \quad {}_L(x) \;\; B^{S2}(\tilde{x})f_L(\tilde{x}) \\
&= [\;\; {}_H(x)f_H(\tilde{x}) + \quad {}_L(x)f_L(\tilde{x})]\left[\frac{{}_H(x)V_H f_H(\tilde{x})}{{}_H(x)f_H(\tilde{x}) + \quad {}_L(x)f_L(\tilde{x})} - B^{S2}(\tilde{x})\right] \\
&= [\;\; {}_H(x)f_H(\tilde{x}) + \quad {}_L(x)f_L(\tilde{x})]\left[\frac{{}_H f_H(x)V_H f_H(\tilde{x})}{{}_H f_H(x)f_H(\tilde{x}) + \quad {}_L f_L(x)f_L(\tilde{x})} - B^{S2}(\tilde{x})\right] \\
&= [\;\; {}_H(x)f_H(\tilde{x}) + \quad {}_L(x)f_L(\tilde{x})]\left[\frac{{}_H V_H \;\; (x) \;\; (\tilde{x})}{{}_H \;\; (x) \;\; (\tilde{x}) + \quad {}_L} - B^{S2}(\tilde{x})\right]; \qquad (37)
\end{aligned}
$$

The rst order condition for bidder 1's surplus maximization problem is

$$\left.\frac{@ \quad {}^{S2}(x;\tilde{x})}{@\tilde{x}}\right|_{\tilde{x}=x} = 0: \qquad (38)$$

Solving for $B^{S2}(x)$, we have

$$B^{S2}(x) = \frac{{}_H V_H \;\; (x)^2}{{}_H \;\; (x)^2 + \quad {}_L}: \qquad (39)$$

The FOC is usually only a necessary condition. It is easy to check that $\frac{{}_H V_H \; (x) \; (\tilde{x})}{{}_H \; (x) \; (\tilde{x}) + \; {}_L}$ is increasing in x. Similar to the argument in Case 1, the surplus function ${}^{S2}(x;\tilde{x})$ is a unimodal function with maximum at $\tilde{x} = x$ when using the bidding function de ned in equation (39). As a result, the su ciency of FOC for the maximization is con rmed.

Again, for this bidding function to be in equilibrium, the condition for \always returning" has to be satis ed. Given that the bidding function is increasing, this condition is equivalent to

$$V_L \le (1 - \quad)B^{S2}(\underline{x}) \qquad (40)$$

$$\Leftrightarrow \quad \le \frac{(V_H - V_L) \;\; {}_H \;\; (\underline{x})^2}{V_H \;\; {}_H \;\; (\underline{x})^2 + \;\; {}_L V_L} \equiv \underline{\quad}^S: \qquad (41)$$

Case 3: Cuto rule when $V = V_L$

In this case, there is an endogenously determined cuto in the winning bidder's return decision. We denote this cuto as x^S. Buyer 1's surplus by pretending to be have signal $\tilde{x}$ is given by

$$
{}^S(x;\tilde{x}) = \begin{cases} {}^{S1}(x;\tilde{x}); & \text{if} \quad \tilde{x} \le x^S; \\ \\ {}_H(x)\left\{\int_{\underline{x}}^{\tilde{x}}[V_H - B^S(x_2)]dF_H(x_2)\right\} & \\ + \;\; {}_L(x)\left\{\int_{\underline{x}}^{x^{S*}}[V_L - B^S(x_2)]dF_L(x_2) - \int_{x^{S*}}^{\tilde{x}} \;\; B^S(x_2)dF_L(x_2)\right\}; & \text{if} \quad \tilde{x} \ge x^S: \end{cases} \qquad (42)
$$

Note that the above function is continuous. Taking the derivative of the above with respect to $\not{x}$, we have

$$\frac{@ \ \ ^S(x;\not{x})}{@\not{x}} = \begin{cases} \frac{@ \ \ ^{S1}(x;\not{x})}{@\not{x}}; & \text{if } \ \not{x} \leq x^S \ ; \\ \frac{@ \ \ ^{S2}(x;\not{x})}{@\not{x}}; & \text{if } \ \not{x} \geq x^S \ : \end{cases} \tag{43}$$

Although $\ ^S(x;\not{x}) \neq \ ^{S2}(x;\not{x})$ when $\not{x} \geq x^S$, we have $\frac{@ \ ^S(x;\not{x})}{@\not{x}} = \frac{@ \ ^{S2}(x;\not{x})}{@\not{x}}$. From the first order condition, we can derive the bidding function as follows:

$$B^S(x) = \begin{cases} B^{S1}(x) = \frac{ _H V_H \ (x)^2 + \ _L V_L}{ _H \ (x)^2 + \ _L}; & \text{if } \ x \leq x^S \ ; \\ B^{S2}(x) = \frac{ _H V_H \ (x)^2}{ _H \ (x)^2 + \ \ _L}; & \text{if } x \geq x^S \ : \end{cases} \tag{44}$$

Note that x^S is determined by

$$(1 - \ )B^{S1}(x^S \ ) = V_L; \tag{45}$$

i.e.,

$$= \frac{(V_H - V_L) \ \ _H \ (x^S \ )^2}{V_H \ \ _H \ (x^S \ )^2 + \ \ _L V_L}: \tag{46}$$

Note that functions $B^{S2}(x)$ and $B^{S1}(x)$ cross each other at x^S .

Now consider the sufficient condition. Given the bidding function (44), from the proof in Cases 1 and 2, we know that $\ ^{S1}(x;\not{x})$ is a unimodal function with the maximum at $\not{x} = x$ when $\not{x} \leq x^S$; and $\ ^{S2}(x;\not{x})$ is a unimodal function with the maximum at $\not{x} = x$ when $\not{x} \geq x^S$. We shall show that $\ ^S(x;\not{x})$ is also a unimodal function with maximum at $\not{x} = x$. Consider $x \leq x^S$, for example. For $\underline{x} \leq \not{x} \leq x$, the payoff is increasing in $\not{x}$ from the first formula of (43). For $x \leq \not{x} \leq x^S$, the payoff is decreasing in $\not{x}$ from the first formula of (43). For $x^S \ \leq \not{x} \leq \overline{x}$, the payoff is decreasing in $\not{x}$ from the second formula of (43). Therefore, $\ ^S(x;\not{x})$ achieves its maximal value at $\not{x} = x$. Similar arguments can be applied to the case of $x \geq x^S$. Thus, the sufficient condition for the maximization is thus satisfied.

In this equilibrium, when $V = V_L$, the winning bidder returns the object if he pays too much, and keeps the object otherwise. For this to happen, has to satisfy the following condition:

$$(1 - \ )B^{S1}(\overline{x}) > V_L > (1 - \ )B^{S1}(\underline{x}) \tag{47}$$

$$\Leftrightarrow \ \underline{\ }^S > \ \ > \overline{\ }^S: \tag{48}$$

Q.E.D.

Proof for Lemma 2

We only need to examine $L_2(\ |x; \ )$ since $L_1(\ |x) = L_2(\ |x; \ = 1)$. Note that the function $\frac{{}_H f_H(x)f_H(s)+ \ {}_L f_L(x)f_L(s)}{{}_H f_H(x)F_H(s)+ \ {}_L f_L(x)F_L(s)}$ is increasing in x. To see this,

$$
\begin{aligned}
& \frac{@\frac{{}_H f_H(x)f_H(s)+ \ {}_L f_L(x)f_L(s)}{{}_H f_H(x)F_H(s)+ \ {}_L f_L(x)F_L(s)}}{@x} \\
= & \frac{@\frac{{}_H f_H(s)+ \ {}_L f_L(s)[\frac{f_L(x)}{f_H(x)}]}{{}_H F_H(s)+ \ {}_L F_L(s)[\frac{f_L(x)}{f_H(x)}]}}{@x} \\
= & \frac{{}_L \ {}_H [f_L(s)F_H(s) - f_H(s)F_L(s)]\frac{@\frac{f_L(x)}{f_H(x)}}{@x}}{\{ \ {}_H F_H(s) + \ {}_L F_L(s)[\frac{f_L(x)}{f_H(x)}]\}^2} \\
= & \frac{{}_L \ {}_H F_L(s)F_H(s)[f_L(s)=F_L(s) - f_H(s)=F_H(s)]\frac{@\frac{f_L(x)}{f_H(x)}}{@x}}{\{ \ {}_H F_H(s) + \ {}_L F_L(s)[\frac{f_L(x)}{f_H(x)}]\}^2} \geq 0 \qquad (49)
\end{aligned}
$$

The inequality follows from the second and third parts of Lemma 1. Therefore,

$$
\begin{aligned}
& \int_{\underline{x}}^{x} \frac{{}_H f_H(s)^2 + \ {}_L f_L(s)^2}{{}_H f_H(s)F_H(s) + \ {}_L f_L(s)F_L(s)} ds & (50) \\
\geq & \int_{\underline{x}}^{x} \frac{{}_H f_H(\underline{x})f_H(s) + \ {}_L f_L(\underline{x})f_L(s)}{{}_H f_H(\underline{x})F_H(s) + \ {}_L f_L(\underline{x})F_L(s)} ds & (51) \\
= & \int_{\underline{x}}^{x} \frac{d ln[\ {}_H f_H(\underline{x})F_H(s) + \ {}_L f_L(\underline{x})F_L(s)]}{ds} ds & (52) \\
= & ln[\ {}_H f_H(\underline{x})F_H(x) + \ {}_L f_L(\underline{x})F_L(x)] - ln[\ {}_H f_H(\underline{x})F_H(\underline{x}) + \ {}_L f_L(\underline{x})F_L(\underline{x})] & (53) \\
= & \infty & (54)
\end{aligned}
$$

Thus $L_2(\underline{x}|x; \ ) = 0$. Moreover $L_2(x|x; \ ) = 1$ and $L_2(\ |x; \ )$ is nondecreasing. As a result, $L_2(\ |x; \ )$ is a distribution function. Q.E.D.

Proof for Proposition 2

Case 1: Never return

Consider a bidder who has signal x and pretends to have $\not{x}$. Given that he always keeps the object if he wins, his expected surplus is given by

$$
\begin{aligned}
{}^{F1}(x; \not{x}) &= Pr(V = V_H|x_1 = x)E\{[V - B^{F1}(\not{x})]I\{x_2 < \not{x}\}|x_1 = x; V = V_H\} & (55) \\
& +Pr(V = V_L|x_1 = x)E\{[V - B^{F1}(\not{x})]I\{x_2 < \not{x}\}|x_1 = x; V = V_L\} & (56) \\
&= \ {}_H(x)[V_H - B^{F1}(\not{x})]F_H(\not{x}) + \ {}_L(x)[V_L - B^{F1}(\not{x})]F_L(\not{x}): & (57)
\end{aligned}
$$

Taking the derivative of the above with respect to $\tilde{x}$, we have

$$
\begin{aligned}
&\frac{\partial \;{}^{F1}(x;\tilde{x})}{\partial \tilde{x}} \\
=\;& {}_H(x)\left[V_H f_H(\tilde{x}) - \left(B^{F1}\right)'(\tilde{x})F_H(\tilde{x}) - B^{F1}(\tilde{x})f_H(\tilde{x})\right] \\
&+ {}_L(x)\left[V_L f_L(\tilde{x}) - \left(B^{F1}\right)'(\tilde{x})F_L(\tilde{x}) - B^{F1}(\tilde{x})f_L(\tilde{x})\right] \\
=\;& \left[{}_H(x)F_H(\tilde{x}) + {}_L(x)F_L(\tilde{x})\right] \times \\
&\times \left\{ \frac{{}_H(x)f_H(\tilde{x}) + {}_L(x)f_L(\tilde{x})}{{}_H(x)F_H(\tilde{x}) + {}_L(x)F_L(\tilde{x})} \left[\frac{V_H \;{}_H(x)f_H(\tilde{x}) + V_L \;{}_L(x)f_L(\tilde{x})}{{}_H(x)f_H(\tilde{x}) + {}_L(x)f_L(\tilde{x})} - B^{F1}(\tilde{x})\right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \\
=\;& \left[{}_H(x)F_H(\tilde{x}) + {}_L(x)F_L(\tilde{x})\right] \times \\
&\times \left\{ \frac{{}_H\;(x)f_H(\tilde{x}) + {}_L f_L(\tilde{x})}{{}_H\;(x)F_H(\tilde{x}) + {}_L F_L(\tilde{x})} \left[\frac{V_H \;{}_H\;(x)\;(\tilde{x}) + V_L \;{}_L}{{}_H\;(x)\;(\tilde{x}) + {}_L} - B^{F1}(\tilde{x})\right] - \left(B^{F1}\right)'(\tilde{x}) \right\}. \qquad (58)
\end{aligned}
$$

The first order condition for this optimization problem is

$$\left.\frac{\partial \;{}^{F1}(x;\tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=x} = 0. \qquad (59)$$

Therefore,

$$\left(B^{F1}\right)'(x) = \frac{{}_H(x)f_H(x) + {}_L(x)f_L(x)}{{}_H(x)F_H(x) + {}_L(x)F_L(x)}\left[\frac{V_H \;{}_H\;(x)^2 + V_L \;{}_L}{{}_H\;(x)^2 + {}_L} - B^{F1}(x)\right]. \qquad (60)$$

The above differential equation (60) is just one of the necessary conditions for the equilibrium. It is also necessary that $\left[\frac{V_H \;{}_H\;(x)^2+V_L\;{}_L}{{}_H\;(x)^2+\;{}_L} - B^{F1}(x)\right]$ is nonnegative; otherwise, bidding zero would be better. Furthermore, $\left[\frac{V_H\;{}_H\;(\underline{x})^2+V_L\;{}_L}{{}_H\;(\underline{x})^2+\;{}_L} - B^{F1}(\underline{x})\right]$ must also be nonpositive; otherwise, the bidder with the lowest signal would be better to bid a little bit more. These last two restrictions determine the boundary condition: $B^{F1}(\underline{x}) = \frac{V_H\;{}_H\;(\underline{x})^2+V_L\;{}_L}{{}_H\;(\underline{x})^2+\;{}_L}$. With this boundary condition and the above differential equation (60), we can obtain the following bidding function:

$$B^{F1}(x) = \int_{\underline{x}}^{x} \frac{{}_H V_H\;(\;)^2 + {}_L V_L}{{}_H\;(\;)^2 + {}_L} dL_1(\;|x); \qquad (61)$$

The bidding function is indeed increasing. Note that this bidding function can also be formulated in the format of Milgrom and Weber [10], i.e., $B^{F1}(x) = \int_{\underline{x}}^{x} E(V|\;;\;)dL_1(\;|x)$.

In what follows, we shall show that $\tilde{x} = x$ indeed maximizes the bidder's surplus given the above bidding function. In equation (58), $\frac{{}_H\;(x)f_H(\tilde{x})+\;{}_L f_L(\tilde{x})}{{}_H\;(x)F_H(\tilde{x})+\;{}_L F_L(\tilde{x})}$ is increasing in x. To see this:

$$
\begin{aligned}
\frac{\partial \frac{{}_H (x) f_H(\not{x}) + {}_L f_L(\not{x})}{{}_H (x) F_H(\not{x}) + {}_L F_L(\not{x})}}{\partial x} &= \frac{{}_L \, {}_H [f_H(\not{x}) F_L(\not{x}) - f_L(\not{x}) F_H(\not{x})] \, '(x)}{[\, {}_H (x) F_H(\not{x}) + {}_L F_L(\not{x})]^2} && (62)\\
&= \frac{{}_L \, {}_H F_L(\not{x}) F_H(\not{x}) [\frac{f_H(\not{x})}{F_H(\not{x})} - \frac{f_L(\not{x})}{F_L(\not{x})}] \, '(x)}{[\, {}_H (x) F_H(\not{x}) + {}_L F_L(\not{x})]^2} && (63)\\
&\geq 0; && (64)
\end{aligned}
$$

where the last inequality is implied from the second part of Lemma 1. Similarly, $\frac{V_H \, {}_H (x) (\not{x}) + V_L \, {}_L}{{}_H (x) (\not{x}) + {}_L}$ is increasing in x. Also note that

$$
\begin{aligned}
B^{F1}(x) &= \int_{\underline{x}}^{x} \frac{{}_H V_H (\;)^2 + {}_L V_L}{{}_H (\;)^2 + {}_L} dL_1(\;|x) && (65)\\
&\leq \int_{\underline{x}}^{x} \frac{{}_H V_H (x)^2 + {}_L V_L}{{}_H (x)^2 + {}_L} dL_1(\;|x) && (66)\\
&= \frac{{}_H V_H (x)^2 + {}_L V_L}{{}_H (x)^2 + {}_L} && (67)
\end{aligned}
$$

Therefore, for $\not{x} < x$, from equation (58),

$$
\begin{aligned}
&\frac{\partial \; {}^{F1}(x; \not{x})}{\partial \not{x}} && (68)\\
= \;& [\, {}_H (x) F_H(\not{x}) + {}_L (x) F_L(\not{x})] \times && (69)\\
& \times \left\{ \frac{{}_H (x) f_H(\not{x}) + {}_L f_L(\not{x})}{{}_H (x) F_H(\not{x}) + {}_L F_L(\not{x})} \left[\frac{V_H \, {}_H (x) (\not{x}) + V_L \, {}_L}{{}_H (x) (\not{x}) + {}_L} - B^{F1}(\not{x}) \right] - \left(B^{F1}\right)' (\not{x}) \right\} && (70)\\
> \;& [\, {}_H (x) F_H(\not{x}) + {}_L (x) F_L(\not{x})] \times && (71)\\
& \times \left\{ \frac{{}_H (x) f_H(\not{x}) + {}_L f_L(\not{x})}{{}_H (x) F_H(\not{x}) + {}_L F_L(\not{x})} \left[\frac{V_H \, {}_H (\not{x}) (\not{x}) + V_L \, {}_L}{{}_H (\not{x}) (\not{x}) + {}_L} - B^{F1}(\not{x}) \right] - \left(B^{F1}\right)' (\not{x}) \right\} && (72)\\
> \;& [\, {}_H (x) F_H(\not{x}) + {}_L (x) F_L(\not{x})] \times && (73)\\
& \times \left\{ \frac{{}_H (\not{x}) f_H(\not{x}) + {}_L f_L(\not{x})}{{}_H (\not{x}) F_H(\not{x}) + {}_L F_L(\not{x})} \left[\frac{V_H \, {}_H (\not{x}) (\not{x}) + V_L \, {}_L}{{}_H (\not{x}) (\not{x}) + {}_L} - B^{F1}(\not{x}) \right] - \left(B^{F1}\right)' (\not{x}) \right\} && (74)\\
= \;& 0 && (75)
\end{aligned}
$$

Similarly, for $\not{x} > x$, from equation (58),

$$
\begin{aligned}
&\frac{\partial \; {}^{F1}(x; \not{x})}{\partial \not{x}} && (76)\\
= \;& [\, {}_H (x) F_H(\not{x}) + {}_L (x) F_L(\not{x})] \times && (77)\\
& \times \left\{ \frac{{}_H (x) f_H(\not{x}) + {}_L f_L(\not{x})}{{}_H (x) F_H(\not{x}) + {}_L F_L(\not{x})} \left[\frac{V_H \, {}_H (x) (\not{x}) + V_L \, {}_L}{{}_H (x) (\not{x}) + {}_L} - B^{F1}(\not{x}) \right] - \left(B^{F1}\right)' (\not{x}) \right\} && (78)\\
< \;& [\, {}_H (x) F_H(\not{x}) + {}_L (x) F_L(\not{x})] \times && (79)
\end{aligned}
$$

$$\times \left\{ \frac{{}_{H}\ (x) f_H(\tilde{x}) + \ {}_{L} f_L(\tilde{x})}{{}_{H}\ (x) F_H(\tilde{x}) + \ {}_{L} F_L(\tilde{x})} \left[\frac{V_H \ {}_{H}\ (\tilde{x})\ (\tilde{x}) + V_L \ {}_{L}}{{}_{H}\ (\tilde{x})\ (\tilde{x}) + \ {}_{L}} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)^{0}(\tilde{x}) \right\} \quad (80)$$

$$< \ [\ {}_{H}(x) F_H(\tilde{x}) + \ {}_{L}(x) F_L(\tilde{x})] \times \quad (81)$$

$$\times \left\{ \frac{{}_{H}\ (\tilde{x}) f_H(\tilde{x}) + \ {}_{L} f_L(\tilde{x})}{{}_{H}\ (\tilde{x}) F_H(\tilde{x}) + \ {}_{L} F_L(\tilde{x})} \left[\frac{V_H \ {}_{H}\ (\tilde{x})\ (\tilde{x}) + V_L \ {}_{L}}{{}_{H}\ (\tilde{x})\ (\tilde{x}) + \ {}_{L}} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)^{0}(\tilde{x}) \right\} \quad (82)$$

$$= \ 0 \quad (83)$$

As a result, the payo function $^{F1}(x; \tilde{x})$ is a unimodal function of $\tilde{x}$ with its maximum at $\tilde{x} = x$, i.e., it is increasing when $\tilde{x} \leq x$ and decreasing when $\tilde{x} \geq x$. Thus, $\tilde{x} = x$ indeed maximizes the bidder's surplus.

Note that for the bidding function to form an equilibrium, the condition of \Never return" must be satis ed. Because the bidding function is increasing, \Never return" is equivalent to

$$(1 - \ ) B^{F1}(\overline{x}) \leq V_L \quad (84)$$

$$\Leftrightarrow \quad \geq \frac{B^{F1}(\overline{x}) - V_L}{B^{F1}(\overline{x})} \quad (85)$$

$$\Leftrightarrow \quad \geq \frac{\int_{\underline{x}}^{\overline{x}} \frac{{}_{H} V_H\ (\ )^2 + \ {}_{L} V_L}{{}_{H}\ (\ )^2 + \ {}_{L}} dL_1(\ |\overline{x}) - V_L}{\int_{\underline{x}}^{\overline{x}} \frac{{}_{H} V_H\ (\ )^2 + \ {}_{L} V_L}{{}_{H}\ (\ )^2 + \ {}_{L}} dL_1(\ |\overline{x})} \quad (86)$$

$$\Leftrightarrow \quad \geq \frac{\int_{\underline{x}}^{\overline{x}} \left\{ \frac{{}_{H} V_H\ (\ )^2 + \ {}_{L} V_L}{{}_{H}\ (\ )^2 + \ {}_{L}} - V_L \right\} dL_1(\ |\overline{x})}{\int_{\underline{x}}^{\overline{x}} \frac{{}_{H} V_H\ (\ )^2 + \ {}_{L} V_L}{{}_{H}\ (\ )^2 + \ {}_{L}} dL_1(\ |\overline{x})} \quad (87)$$

$$\Leftrightarrow \quad \geq \frac{\int_{\underline{x}}^{\overline{x}} \frac{(V_H \ V_L)\ {}_{H}\ (\ )^2}{V_H \ {}_{H}\ (\ )^2 + \ {}_{L}} dL_1(\ |\overline{x})}{\int_{\underline{x}}^{\overline{x}} \frac{{}_{H} V_H\ (\ )^2 + \ {}_{L} V_L}{{}_{H}\ (\ )^2 + \ {}_{L}} dL_1(\ |\overline{x})} = {}^{-F}. \quad (88)$$

Case 2: Always return when $V = V_L$

In this case, the winning bidder always returns the object when $V = V_L$. A bidder's surplus when having signal x but pretending to be $\tilde{x}$ is given by

$$^{F2}(x; \tilde{x}) = Pr(V = V_H | x_1 = x) E\{(V - B^F(\tilde{x})) I\{x_2 < \tilde{x}\} | x_1 = x; V = V_H\} \quad (89)$$

$$-Pr(V = V_L | x_1 = x) E\{\ B^F(\tilde{x}) I\{x_2 < \tilde{x}\} | x_1 = x; V = V_L\} \quad (90)$$

$$= \ {}_{H}(x)[V_H - B^F(\tilde{x})] F_H(\tilde{x}) - \ {}_{L}(x)\ B^F(\tilde{x}) F_L(\tilde{x}). \quad (91)$$

Taking the derivative of the above with respect to $\tilde{x}$, we have

$$\frac{@\ ^{F2}(x; \tilde{x})}{@\tilde{x}}$$

$$= \ {}_{H}(x)[V_H f_H(\tilde{x}) - B^{F2\,\prime}(\tilde{x}) F_H(\tilde{x}) - B^{F2}(\tilde{x}) f_H(\tilde{x})]$$
$$+ \ {}_{L}(x)[- \ B^{F2\,\prime}(\tilde{x}) F_L(\tilde{x}) - \ B^{F2}(\tilde{x}) f_L(\tilde{x})]$$

$$
\begin{aligned}
&= \left[\pi_H(x)F_H(\tilde{x}) + \pi_L(x)F_L(\tilde{x})\right] \times \\
&\quad \times \left\{ \frac{\pi_H(x)f_H(\tilde{x}) + \pi_L(x)f_L(\tilde{x})}{\pi_H(x)F_H(\tilde{x}) + \pi_L(x)F_L(\tilde{x})} \left[\frac{V_H \pi_H(x)f_H(\tilde{x})}{\pi_H(x)f_H(\tilde{x}) + \pi_L(x)f_L(\tilde{x})} - B^{F2}(\tilde{x}) \right] - \left(B^{F2}\right)'(\tilde{x}) \right\} \\
&= \left[\pi_H(x)F_H(\tilde{x}) + \pi_L(x)F_L(\tilde{x})\right] \times \\
&\quad \times \left\{ \frac{\pi_H\lambda(x)f_H(\tilde{x}) + \pi_L f_L(\tilde{x})}{\pi_H\lambda(x)F_H(\tilde{x}) + \pi_L F_L(\tilde{x})} \left[\frac{V_H \pi_H \lambda(x)\lambda(\tilde{x})}{\pi_H\lambda(x)\lambda(\tilde{x}) + \pi_L} - B^{F2}(\tilde{x}) \right] - \left(B^{F2}\right)'(\tilde{x}) \right\}. \qquad (92)
\end{aligned}
$$

The first order condition for this optimization problem is

$$\left.\frac{\partial \Pi^{F2}(x;\tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=x} = 0. \qquad (93)$$

Therefore,

$$\left(B^{F2}\right)'(x) = \frac{\pi_H(x)f_H(x) + \pi_L(x)f_L(x)}{\pi_H(x)F_H(x) + \pi_L(x)F_L(x)} \left[\frac{V_H \pi_H \lambda(x)^2}{\pi_H\lambda(x)^2 + \pi_L} - B^{F2}(x) \right]. \qquad (94)$$

This differential equation (94) is just one necessary condition for the equilibrium. It must also be that $\left[\frac{V_H \pi_H \lambda(x)^2}{\pi_H\lambda(x)^2+\pi_L} - B^{F2}(x)\right]$ be nonnegative; otherwise, bidding zero would be better. Also, $\left[\frac{V_H \pi_H \lambda(\underline{x})^2}{\pi_H\lambda(\underline{x})^2+\pi_L} - B^{F2}(\underline{x})\right]$ must be nonpositive; otherwise, the bidder with the lowest signal would be better bidding a little bit more. These two conditions determine the boundary condition: $B^{F2}(\underline{x}) = \frac{V_H \pi_H \lambda(\underline{x})^2}{\pi_H\lambda(\underline{x})^2+\pi_L}$. With this boundary condition, we obtain the following bidding function from the differential equation (94):

$$B^{F2}(x) = \int_{\underline{x}}^{x} \frac{V_H \pi_H \lambda(\alpha)^2}{\pi_H\lambda(\alpha)^2 + \pi_L} dL_2(\alpha|x;\beta), \qquad (95)$$

The bidding function is indeed increasing.

In what follows, we shall show that $\tilde{x} = x$ indeed maximizes the bidder's surplus given the above bidding function. In equation (92), it is easy to see that $\frac{\pi_H\lambda(x)f_H(\tilde{x})}{\pi_H\lambda(x)F_H(\tilde{x})+\pi_L F_L(\tilde{x})}$ and $\frac{V_H\pi_H\lambda(x)\lambda(\tilde{x})+V_L\pi_L}{\pi_H\lambda(x)\lambda(\tilde{x})+\pi_L}$ are both increasing in x. Following a similar argument as in Case 1, we can show that the payoff function $\Pi^{F2}(x;\tilde{x})$ is a unimodal function of $\tilde{x}$ with its maximum at $\tilde{x} = x$ given the bidding function, i.e., it is increasing when $\tilde{x} \leq x$ and decreasing when $\tilde{x} \geq x$. Therefore, $\tilde{x} = x$ indeed maximizes the bidder's payoff.

This equilibrium bidding function is based on the condition that the winning bidder always returns the object if $V = V_L$, which is equivalent to

$$(1-\delta)B^{F2}(\underline{x}) \geq V_L, \qquad (96)$$

$$\Leftrightarrow \quad (1 - \quad) \int_{\underline{x}}^{x} \frac{V_H \quad _H \; (\;)^2}{_H \; (\;)^2 + \quad _L} dL_2(\; | \underline{x}; \;) \geq V_L \tag{97}$$

$$\Leftrightarrow \quad (1 - \quad) \frac{V_H \quad _H \; (\underline{x})^2}{_H \; (\underline{x})^2 + \quad _L} \geq V_L \tag{98}$$

$$\Leftrightarrow \quad \leq \frac{(V_H - V_L) \quad _H \; (\underline{x})^2}{V_H \quad _H \; (\underline{x})^2 + \; _L V_L} = \underline{\;}^F : \tag{99}$$

Case 3: Cuto rule when $V = V_L$

In this case, there is an endogenously determined cuto in the winning bidder's return decision. We denote this cuto as x^F . A bidder 1's surplus by pretending to be have signal $\tilde{x}$ is given by

$$^F(x; \tilde{x}) = \begin{cases} ^{F1}(x; \tilde{x}); & \text{if } \tilde{x} \leq x^F \; ; \\ ^{F2}(x; \tilde{x}); & \text{if } \tilde{x} \geq x^F \; : \end{cases} \tag{100}$$

Taking the derivative with respect to $\tilde{x}$, we have

$$\frac{@ \quad ^F(x; \tilde{x})}{@\tilde{x}} = \begin{cases} \frac{@ \quad ^{F1}(x; \tilde{x})}{@\tilde{x}}; & \text{if } \tilde{x} \leq x^F \; ; \\ \\ \frac{@ \quad ^{F2}(x; \tilde{x})}{@\tilde{x}}; & \text{if } \tilde{x} \geq x^F \; : \end{cases} \tag{101}$$

Note that the above payo function is continuous.

If $x \leq x^F$, the necessary condition for the optimization implies that $B^F(x) = B^{F1}(x)$. This bidding function also pins down the cuto x^F , which is determined by $B^{F1}(x^F \;) = \frac{V_L}{1}$. For $x \geq x^F$, the rst order condition is again given by equation (92), but the initial condition is di erent and is replaced by $B^{F2}(x^F \;) = \frac{V_L}{1}$. Solving (92) with this initial condition, we have

$$B^F(x) = \int_{\underline{x}}^{x} \frac{V_H \quad _H \; (\;)^2}{_H \; (\;)^2 + \quad _L} dL_2(\; |x; \;) + AL_2(x^F \; |x; \;) \tag{102}$$

$$= B^{F1}(x) + AL_2(x^F \; |x; \;); \tag{103}$$

where

$$A = \frac{V_L}{1 -} - \int_{\underline{x}}^{x^{F*}} \frac{_H V_H \; (\;)^2}{_H \; (\;)^2 + \quad _L} dL_2(\; |x^F \; ; \;): \tag{104}$$

To summarize, the equilibrium bidding function in this case is characterized by the following:

$$B^F(x) = \begin{cases} B^{F1}(x); & \text{if } x \leq x^F \ ; \\ B^{F2}(x) + AL_2(x^F \ |x; \ ); & \end{cases}$$

and make use of (110). We obtain

$$
\begin{aligned}
{}^{0}(x) &= M_1^A(x;x) - M_1^B(x;x) + M_2^A(x;x) - M_2^B(x;x) && (113)\\
&= V_H \;\; _H(x) f_H(x) - V_H \;\; _H(x) f_H(x) + M_2^A(x;x) - M_2^B(x;x) && (114)\\
&= M_2^A(x;x) - M_2^B(x;x) \geq 0: && (115)
\end{aligned}
$$

The inequality above is by assumption. Since $(\underline{x}) = 0$, we conclude that for all x, $(x) \geq 0:$
Q.E.D.

Proof for Proposition 4

When $V_L = 0$, in the second-price auctions,

$$M^{S2}(x; x; \;) = \; _H(x) \int^{x)} \;) =$$

It is obvious that $K(\ ;x;\ )$ is decreasing in .

$$\frac{@L_2(\ |x;\ )}{@} = -L_2(\ |x;\ )\int^x \frac{{}_L f_L(s)^2\ {}_H f_H(s)F_H(s) - \ {}_H f_H(s)^2\ {}_L f_L(s)F_L(s)}{[\ {}_H f_H(s)F_H(s) + \ {}_L f_L(s)F_L(s)]^2}ds$$

$$= -L_2(\ |x;\ )\int^x \frac{{}_L\ {}_H f_L(s)^2 f_H(s)^2[\frac{F_H(s)}{f_H(s)} - \frac{F_L(s)}{f_L(s)}]}{[\ {}_H f_H(s)F_H(s) + \ {}_L f_L(s)F_L(s)]^2}ds \geq 0: \quad (123)$$

From Lemma 2, we know that $L_2(\ |x;\ )$ is a distribution. This means that, for $_1 \leq\ _2$, $L_2(\ |x;\ _1)$ rst order stochastic dominates $L_2(\ |x;\ _2)$. De ne $= \frac{F_H(x)}{F_L(x)}$. For $_1 \leq\ _2 \leq$,

$$M_2^{F2}(x;x;\ _1) = {}^0_H(x)\int_{\underline{x}}^x K(\ ;x;\ _1)dL_2(\ |x;\ _1)$$

$$\geq {}^0_H(x)\int_{\underline{x}}^x K(\ ;x;\ _2)dL_2(\ |x;\ _1) \quad (124)$$

$$\geq {}^0_H(x)\int_{\underline{x}}^x K(\ ;x;\ _2)dL_2(\ |x;\ _2) \quad (125)$$

$$= M_2^{F2}(x;x;\ _2): \quad (126)$$

The rst inequality holds because $K(\ ;x;\ )$ is decreasing in . The second inequality holds because $K(\ ;x;\ )$ is increasing in and $L_2(\ |x;\ _1)$ rst order stochastic dominates $L_2(\ |x;\ _2)$. Thus $M_2^{F2}(x;x;\ ))$ is decreasing in for $\leq$.

For $\geq$

$$M_2^{F2}(x;x;\ ) \quad (127)$$

$$= {}^0_H(x)\int_{\underline{x}}^x \frac{{}_H V_H\ (\ )^2[F_H(x) - \ F_L(x)]}{{}_H\ (\ )^2 + \ {}_L} \frac{{}_H f_H(\ )^2 + \ {}_L f_L(\ )^2}{{}_H f_H(\ )F_H(\ ) + \ {}_L f_L(\ )F_L(\ )} L_2(\ |x;\ )d$$

$$= {}^0_H(x)\int_{\underline{x}}^x \underbrace{\frac{{}_H V_H f_H(\ )^2[F_H(x) - \ F_L(x)]}{{}_H f_H(\ )F_H(\ ) + \ {}_L f_L(\ )F_L(\ )}}_{\text{denote as } Q(\ ;x;\ )} L_2(\ |x;\ )d\ : \quad (128)$$

$$\frac{@M_2^{F2}(x;x;\ )}{@} = {}^0_H(x)\int_{\underline{x}}^x \left[\underbrace{\frac{@Q(\ ;x;\ )}{@}}_{0} \underbrace{L_2(\ |x;\ )}_{0} + \underbrace{Q(\ ;x;\ )}_{0} \underbrace{\frac{@L_2(\ |x;\ )}{@}}_{0}\right] d\ \leq 0: \quad (129)$$

Hence, $M_2^{F2}(x;x;\ )$ is decreasing in for $\geq$. As a result, $M_2^{F2}(x;x;\ )$ is decreasing in all the time. Therefore, from Proposition 3, the seller's revenue is decreasing in . Q.E.D.

Proof for Proposition 6

*S*2

$$-\int_{\underline{x}}^{x} \frac{{}_H V_H\ (\ )^2}{{}_H\ (\ )^2 + {}_L} dL_2(\ |x;\ )\Big\}: \quad (142)$$

It is straightforward to show that $W(\ ;x;\ )$ can be regarded as a distribution for on $[\underline{x};x]$. Furthermore, since $\frac{{}_H f_H(x) f_H(s) + {}_L f_L(x) f_L(s)}{{}_H f_H(x) F_H(s) + {}_L f_L(x) F_L(s)}$ is increasing in x from (49), we have

$$\begin{aligned}
L_2(\ |x;\ ) &= e^{\int^{x} \frac{{}_H f_H(s)^2 + {}_L f_L(s)^2}{{}_H f_H(s) F_H(s) + {}_L f_L(s) F_L(s)} ds} && (143)\\
&\geq e^{\int^{x} \frac{{}_H f_H(x) f_H(s) + {}_L f_H(x) f_L(s)}{{}_H f_H(x) F_H(s) + {}_L f_L(x) F_L(s)} ds} && (144)\\
&= e^{\int^{x} \frac{d\ln[{}_H f_H(x) F_H(s) + {}_L f_L(x) F_L(s)]}{ds} ds} && (145)\\
&= e^{\ln[{}_H f_H(x) F_H(\ ) + {}_L f_L(x) F_L(\ )]\ \ln[{}_H f_H(x) F_H(x) + {}_L f_L(x) F_L(x)]} && (146)\\
&= \frac{{}_H f_H(x) F_H(\ ) + {}_L f_L(x) F_L(\ )}{{}_H f_H(x) F_H(x) + {}_L f_L(x) F_L(x)} && (147)\\
&= W(\ ;x;\ ): && (148)
\end{aligned}$$

Thus, $W(\ ;x;\ )$ rst order stochastic dominates $L_2(\ |x;\ )$. Since $\frac{{}_H V_H\ (\ )^2}{{}_H\ (\ )^2 + {}_L}$ is an increasing function of , we can conclude that $M^{S2}(x;x;\ ) \geq M^{F2}(x;x;\ )$. Q.E.D.

Proof for Proposition 7

Since the linkage principle cannot be applied for revenue ranking, we will do a direct comparison. Below, we examine the seller's revenue in the second-price auctions in three cases. Let $R^S(\ )$ denote the seller's expected revenue as a function of .

Case 1: $\geq \bar{\ }^S$:

In this case, the seller's revenue does not depend on :

$$\frac{1}{2}R^S(\ ) \quad (149)$$

$$= {}_H \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S1}(x_2) f_H(x_1) f_H(x_2) dx_1 dx_2 + {}_L \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S1}(x_2) f_L(x_1) f_L(x_2) dx_1 dx_2$$

$$= \int_{\underline{x}}^{\overline{x}} \frac{{}_H V_H\ (x_2)^2 + {}_L V_L}{{}_H\ (x_2)^2 + {}_L} [{}_H(1 - F_H(x_2)) f_H(x_2) + {}_L(1 - F_L(x_2)) f_L(x_2)] dx_2: \quad (150)$$

Case 2: $\leq \underline{\ }^S$.

$$\frac{1}{2}R^S(\ ) \quad (151)$$

$$= {}_H \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S2}(x_2) f_H(x_1) f_H(x_2) dx_1 dx_2 + {}_L \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S2}(x_2) f_L(x_1) f_L(x_2) dx_1 dx_2$$

$$+ {}_L \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} V_0 f_L(x_1) f_L(x_2) dx_1 dx_2 \quad (152)$$

$$= \quad {}_{H}\int_{\underline{x}}^{\overline{x}} B^{S2}(x_2)[1-F_H(x_2)]f_H(x_2)dx_2 + \quad {}_{L}\int_{\underline{x}}^{\overline{x}} \quad B^{S2}(x_2)[1-F_L(x_2)]f_L(x_2)dx_2 \tag{153}$$

$$+\frac{{}_{L}V_0}{2}: \tag{154}$$

Therefore,

$$\frac{1}{2}(R^S)^{0}(\quad) \tag{155}$$

$$= \quad {}_{H}\int_{\underline{x}}^{\overline{x}} \frac{@B^{S2}(x_2)}{@}[1-F_H(x_2)]f_H(x_2)dx_2 + \quad {}_{L}\int_{\underline{x}}^{\overline{x}} \frac{@[\quad B^{S2}(x_2)]}{@}[1-F_L(x_2)]f_L(x_2)dx_2$$

$$= \int_{\underline{x}}^{\overline{x}} \Big\{ \frac{{}_{H}V_H \quad (x_2)^2 \quad {}_{L}}{-[\quad {}_{H} \quad (x_2)^2 + \quad {}_{L}]^2} \quad {}_{H}[1-F_H(x_2)]f_H(x_2) \tag{156}$$

$$+ \frac{{}_{H}V_H \quad (x_2)^2 \quad {}_{H} \quad (x_2)^2}{[\quad {}_{H} \quad (x_2)^2 + \quad {}_{L}]^2} \quad {}_{L}[1-F_L(x_2)]f_L(x_2)\Big\}dx_2 \tag{157}$$

$$= \int_{\underline{x}}^{\overline{x}} \frac{{}_{H}V_H \quad (x_2)^2 \quad {}_{H} \quad {}_{L}}{[\quad {}_{H} \quad (x_2)^2 + \quad {}_{L}]^2}\Big\{ \quad (x_2)^2[1-F_L(x_2)]f_L(x_2) - [1-F_H(x_2)]f_H(x_2)\Big\}dx_2 \tag{158}$$

$$= \int_{\underline{x}}^{\overline{x}} \frac{{}^{2}_{H} \quad {}_{L}V_H \quad (x_2)^2 f_H(x_2)^2}{[\quad {}_{H} \quad (x_2)^2 + \quad {}_{L}]^2}\Big\{\frac{1-F_L(x_2)}{f_L(x_2)} - \frac{1-F_H(x_2)}{f_H(x_2)}\Big\}dx_2 \le 0 \tag{159}$$

The inequality follows the rst part of Lemma 1. Thus, $R^{0}(\quad) \le 0$; the seller's revenue is decreasing in .

Case 3: $\underline{\quad}^{S} < \quad < \overline{\quad}^{S}$.

In this case,

$$\frac{1}{2}R^S(\quad) \tag{160}$$

$$= \quad {}_{H}\Big\{\int_{\underline{x}}^{x^{S*}} B^{S1}(x_2)[1-F_H(x_2)]f_H(x_2)dx_2 + \int_{x^{S*}}^{\overline{x}} B^{S2}(x_2)[1-F_H(x_2)]f_H(x_2)dx_2\Big\}$$

$$+ \quad {}_{L}\Big\{\int_{\underline{x}}^{x^{S*}} B^{S1}(x_2)[1-F_L(x_2)]f_L(x_2)dx_2 + \int_{x^{S*}}^{\overline{x}} \quad B^{S2}(x_2)[1-F_L(x_2)]f_L(x_2)dx_2\Big\}$$

$$+ \quad {}_{L}\int_{x^{S*}}^{\overline{x}} V_0[1-F_L(x_2)]f_L(x_2)dx_2; \tag{161}$$

Note that x^{S} is also a function of .

$$\frac{1}{2}(R^S)^{0}(\quad)$$

$$= \quad {}_{H}\underbrace{[B^{S1}(x^{S}\quad) - B^{S2}(x^{S}\quad)]}_{=0}[1-F_H(x^{S}\quad)]f_H(x^{S}\quad)\frac{dx^S}{d}$$

$$+ \quad {}_{H}\int_{x^{S*}}^{\overline{x}} \frac{@B^{S2}(x_2)}{@}[1-F_H(x_2)]f_H(x_2)dx_2 + \quad {}_{L}\int_{x^{S*}}^{\overline{x}} \frac{@[\quad B^{S2}(x_2)]}{@}[1-F_L(x_2)]f_L(x_2)dx_2$$

$$+ \quad _L [B^{S1}(x^S \ ) - \ B^{S2}(x^S \ )]$$

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