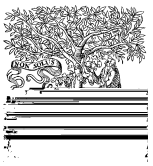

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Communication networks with endogenous link strength

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Received 23 November 2005

Abstract

This paper analyzes the formation of networks when players choose how much to invest in each relationship. We suppose that players have a fixed endowment that they can allocate across links, and in the baseline model, suppose that link strength is an additively separable and convex function of individual investments, and that agents use the path which maximizes the product of link strengths. We show that both the stable and efficient network architectures are stars. However, the investments of the hub may differ in stable and efficient networks. Under alternative assumptions on the investment technology and the reliability measure, other network architectures can emerge as efficient and stable.

JEL classification: D85; C70

Keywords: Communication networks; Network reliability; Endogenous link strength

1. Introduction

Following a long tradition in sociology, economists have recently focussed attention on the role of social networks in economic activities. One of the main contributions of this emerging literature has been to propose strategic models of network formation, where self-interested agents establish bilateral links in order to maximize their utility. Following the pioneering work of Bala and Goyal (2000) and Jackson and Wolinsky (1996), most of the literature assumes that agents make a discrete decision—namely, choose whether or not to invest in a link of fixed quality. However, in a wide variety of contexts arising in both formal and informal networks, agents do not only choose with whom they link, but also how much they spend on every link they form. In this paper, our objective is to study a model of network formation, where the quality of links is endogenously chosen by the agents.[2]

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[1] Also affiliated with the University of Warwick.
[2] See Goyal (2005), which also emphasizes the importance of studying networks where the strength of links can be chosen endogenously.

0899-8256/$ – see front matter
doi:10.1016/j.geb.2008.03.007

Please cite this article in press as: F. Bloch, B. Dutta, Communication networks with endogenous link strength, Games Econ. Behav. (2008), doi:10.1016/j.geb.2008.03.007

Our analysis is centered around communication networks—networks where agents derive positive benefits from the agents with whom they are connected, with benefits decreasing as the distance increases between two agents. Communication networks can either be formal networks (like the phone or Internet), or informal networks of social relations.[3] In formal communication networks, the reliability of a communication link depends on the physical characteristics of the connection; in informal social networks, the strength of a social link depends on the frequency and the length of social interactions. In both cases, it is commonly observed that different links may have different quality, and that agents can choose the amount they invest on every relation. Communication networks thus represent an obvious testing ground for a theory of network formation with endogenous link strength.

The modeling of network formation with endogenous link strength poses new conceptual difficulties, which were absent in the literature where links have a fixed, exogenous value. First, the technology transforming individual investments into the quality of a bilateral link needs to be specified. In this exploratory analysis of the formation of links with endogenous quality, we focus attention on the simple situation where agents' investment decisions are independent. We also assume that agents may face a fixed cost in the formation of links. This leads us to specify the strength of a link as an *additively separable and convex function of agents' investments*. In this setting, agents do not need the consent of their partner to form a link, and the model can be interpreted as a generalization of Bala and Goyal's (2000) version of one-sided, two-way flow model of link formation. In a later Section of the paper, we also provide a partial analysis of the case of *perfect complements*, where the strength of a link is given by the minimum of both parties' investments. This case is reminiscent of Jackson and Wolinsky's (1996) model of link formation with consent, and our analysis generalizes their approach to handle weighted networks where agents optimally "match" investments on their bilateral link.

The second difficulty stems from the modeling of benefits from *indirect* connections. In the existing literature, it is assumed that agents always choose to connect through the shortest path, and that indirect benefits are a decreasing function of the length of the connection.[4] When link strength is endogenous, it is natural to suppose that agents choose to connect through the path with the highest reliability (measured by the product of link strengths normalized to belong to $(0, 1)$)—which is not necessarily the shortest path. However, this is not the only way to model the reliability of an indirect connection. At the end of the paper, we consider an alternative model, where the value of an indirect connection depends on the weakest link along the path.

Throughout the paper, we consider the following problem. We suppose that agents are endowed with a fixed endowment X, that they allocate across different connections. Our first task is to compute the strongly efficient network architecture, which maximizes the sum of benefits of all agents. In a second step, we characterize the set of *stable* networks, obtained when agents form relations in a voluntary, decentralized manner. Because of the well-known coordination problem in noncooperative games of bilateral link formation, we consider two equilibrium concepts. We say that a network is *Nash stable* if it is immune to *individual* deviations, and *strongly pairwise stable* if it is immune to deviations by individuals as well as *pairs* of agents.

The main results of our study can be summarized as follows. We first show that the efficient network must be a *star*, that is a network where one agent (the hub) is connected to all other agents, while peripheral agents are only connected with the hub. Moreover, when link strength is a linear function of individual investments, the unique efficient network

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weakest-link reliability, where the value of an indirect connection is independent of the distance, and is only affected by the strength of the weakest link along the path. In this case, the symmetric star again emerges as an efficient network architecture.[5] The symmetric star is also strongly pairwise stable, as any deviation from a network where all links have equal strength is bound to decrease the value of the weakest link.

Second, we consider the case where individual investments act as perfect complements in the "production" of link strength. The efficient network architecture is now very different. Trees cannot be optimal, because end players of the network waste part of their endowments, and have an incentive to invest what remains from their endowment on a new link. Regular graphs (where all agents have the same number of links) are likely candidates for efficient network architectures, and we show that for small numbers of players, the circle is in fact the efficient network architecture. However, when the number of players increases, the circle can be dominated by other network architectures. Finally, we note that the circle is always strongly pairwise stable.

Our results are related to the results obtained in the discrete link formation literature [Bala and Goyal (2000) two-way flow model with decay, Hojman and Szeidl (2008) model with strong decreasing returns to scale and decay, and Feri's (2007) evolutionary model] and we now comment on the relation between our analyses. Bala and Goyal (2000) also characterize the star as the efficient network architecture in their two-way flow model with decay when the cost parameter is such that neither the empty nor the complete graph are efficient (Bala and Goyal, 2000, Proposition 5.5, p. 1220). However, Nash equilibrium has little predictive power in their model, and even when they resort to the refinement of strict Nash equilibria, they are unable to obtain a complete characterization in the two-way flow model with decay (see Bala and Goyal, 2000, Proposition 5.3 p. 1215).) By placing additional restrictions on the benefit function and the effect of decay, Hojman and Szeidl (2008) are able to fully characterize the set of Nash equilibria, and show that it is a periphery-sponsored star (Hojman and Szeidl, 2008, Theorem 1). However, the characterization of efficient networks in their model is complicated (see the Example 1 and Proposition 1 in Hojman and Szeidl, 2008). Feri (2007) also characterizes periphery-sponsored stars as the unique stochastically stable equilibria of his model.

The similarity between these characterizations and our results are partly driven by our assumption on the link formation technology. Because the formation of links typically involves a fixed cost, we find natural to assume that link strength is a convex function of individual investments. As a consequence, agents have an incentive to concentrate their investments on a single link, and the model appears to be similar to a model of discrete link formation, where every agent forms a single link of maximal intensity. However, there remain significant differences between our model of endogenous link quality and discrete link formation models. First, because we consider a richer set of weighted networks, agents have more opportunities. To establish that the star is the efficient network in our setting is a much harder task than in Bala and Goyal's (2000) analysis, because we are optimizing over a much larger set of feasible networks. Similarly, in the noncooperative game of link formation, we characterize best responses in a larger space of strategies. Second, in our setting, the fixed cost can be made arbitrarily low, and we devote much attention to the case of linear investments where the fixed cost is equal to zero. With linear investments, agents have no a priori reason to concentrate their investments on a single link, and the emergence of equilibrium networks where some agents invest all their resources in a single link is not driven by the assumption on technology but by the general structure of the model. Finally, in our analysis, the concentration of investments in a single link is an equilibrium result rather than an assumption. As we will argue below, the fact that agents *could have* invested in multiple links changes the analysis deeply, and is the driving force behind our sharp characterization of Nash and strongly pairwise equilibria of the game of link formation.

1.1. Related work

Given the obvious importance of networks with links of varying strength, a number of recent papers have proposed models where agents choose how much to invest in a relationship. In some of these models, agents choose their investment *after* the network has been established. For example, Bramoulle and Kranton (2007) study the agent's incentives to provide a public good once the network is fixed. In a specific model of strategic alliances among firms, Goyal and Moraga Gonzales (2001) consider a two-stage model where firms first form links and then decide their R&D investment. This is a model of "nonspecific networking" because the firm chooses the same investment across

[5] However, this is not the unique efficient architectures—other networks which generate the same distribution of link strengths are equally efficient.

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all its links. Durieu et al. (2004) also consider a model of nonspecific networking. Agents choose a single investment, which applies to the links with all other agents. Still in the framework of nonspecific networking, Cabrales et al. (2007) propose an axiomatic derivation of the relation between pairwise link intensities and agents' "socialization intensities", represented by scalars. Brueckner (2003) considers a model of friendship networks. Agents choose to invest in relationships, and the value of indirect benefits is given by the product of the strength of links. For most of his analysis, Brueckner (2003) concentrates on *three* player networks, and studies the effect of the network structure on the investment choices in the complete and star networks. Other papers, more closely related to ours, consider the formation of the network and the choice of investments as simultaneous. Goyal et al. (in press) extend their analysis of cost-reducing alliances by allowing firms to choose different investments on different links. Rogers (2005) proposes a different model of network formation with endogenous link quality. In his model, links are directed and can be interpreted as the influence that every agent has on another agent. As in our paper, agents allocate a fixed endowment on different relationships. Agent's utilities depend on the values of other agents in their neighborhoods, and are defined in a circular way. Two different models are studied: one where agents receive value from their neighbors, and one where they give values to their neighbors. In this environment, which is very different from ours, Rogers (2005) characterizes Nash and efficient network structures, emphasizing the importance of heterogeneity across agents.

2. Model and notations

2.1. Investments and link strength

Let $N = \{1, 2, \ldots, n\}$ be a set of individuals. Individuals derive benefits from links to other individuals. These benefits may be the pleasure from friendship, or the utility from (non-rival) information possessed by other individuals, and so on. In order to fix ideas, we will henceforth interpret benefits as coming from information possessed by other individuals. Each individual has a total resource (time, money) of $X > 0$, and has to decide on how to allocate X in establishing links with others.[6]

Let x_i^j denote the amount of resource invested by player i in the relationship with j. Then, the *strength* of the relationship between i and j, s_{ij} is assumed to be a symmetric, additively separable function of x_i^j and x_j^i,

$$s_{ij} = \phi\left(x_i^j\right) + \phi\left(x_j^i\right)$$

where $\phi(.)$ is a nondecreasing, convex function. Furthermore, we suppose that $\phi(0) = 0$ and $\phi(X) < 1/2$ so that $s_{ij} \in (0, 1)$.

Some remarks are in order. First, we consider a setting where link strength is an additively separable function of investments. This implies that an agent's decision to allocate his endowment over *direct links* is independent of his neighbors' decisions. However, this does not mean that an agent's investment strategy is independent of the choices of other agents, as these choices affect the value of *indirect links* and hence the payoffs obtained in the game. Second,

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Given any g, a path between individuals i and j is a sequence $i^0 = i, i^1, \ldots, i^m, \ldots, i^M = j$ such that $i^{m+1}i^m \in g$ for all $m \in \{0, \ldots, M-1\}$. Two individuals are *connected* if there exists a path between them. Connectedness defines an equivalence relation, and we can partition the set of individuals according to this relation. Blocks of that partition are called components.

Suppose i and j are connected. Then, the benefit that i derives from j depends on the *reliability* with which i can access j's information. There are different ways to model how the strength of links affects the reliability of the communication channel. In our view, the most natural interpretation is that the strength of a link is an index of the quality of the transmission, so that messages sent along stronger links are more likely to be delivered without delay or distortion. With this interpretation, the reliability of any path between i and j is given by the product of the link strengths along the path. For any path $p(i, j) = i, i^1, \ldots, i^{M-1}, j$, we thus define

$$r\big(p(i,j)\big) = s_{ii^1} \ldots s_{i^{m-1}i^m} \ldots s_{i^{M-1}j}.$$

We assume that agents always choose to transmit information along the path with the highest reliability, and for any connected pair of agents i, j, we let $p^*(i, j)$ denote the most reliable path in the set of all paths $P(i, j)$

$$p^*(i,j) = \arg \max_{p(i,j) \in P(i,j)} r\big(p(i,j)\big).$$

The benefit of the connection from i to j is then given by

$$R(i,j) = r\big(p^*(i,j)\big) = \max_{p(i,j) \in P(i,j)} r\big(p(i,j)\big).$$

The total utility that agent i obtains in the weighted graph g can then be computed as:

$$U_i(g) = \sum_{j \neq i} R(i,j),$$

and the total value of the graph is given by

$$V(g) = \sum_i U_i(g).$$

2.3. Efficient and stable networks

We now define efficient and stable graphs. The notion of efficiency that we use is the strong efficiency notion introduced by Jackson and Wolinsky (1996).

Definition 1. A graph g is efficient if $V(g) \geqslant V(g')$ for all g'.

We now describe the concepts of stability that will be used in this paper.

Given any pattern of investments $\mathbf{x}$, and individual i, $(\mathbf{x}_{-\mathbf{i}}, x_i')$ denotes the vector where i deviates from x_i to x_i'. Similarly, $(\mathbf{x}_{-\mathbf{i},\mathbf{j}}, x_{i,j}')$ denotes the vector where i and j have jointly deviated from (x_i, x_j) to (x_i', x_j').

Definition 2. A graph $g(\mathbf{x})$ is Nash stable if there is no individual i and x_i' such that $U_i(g(\mathbf{x}_{-\mathbf{i}}, x_i')) > U_i(g(\mathbf{x}))$.

So, a graph g induced by a vector $\mathbf{x}$ is Nash stable if no individual can change her pattern of investment in the different links and obtain a higher utility.

Definition 3. A graph $g(\mathbf{x})$ is strongly pairwise stable if it is Nash stable and there is no pair of individuals (i, j) and joint deviation (x_i', x_j') such that

$$U_k\big(g(\mathbf{x}_{-\mathbf{i},\mathbf{j}}, x_{i,j}')\big) > U_k\big(g(\mathbf{x})\big) \quad \text{for } k = i, j.$$

A Nash-stable graph is strongly pairwise stable if no pair of individuals can both be strictly better off by changing their pattern of investment. Jackson and Wolinsky (1996) define a weaker notion of stability—*pairwise stability*.

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They basically restrict deviations by assuming that only one link at a time can be changed. Our current definition corresponds to the definition of pairwise stability used by Dutta and Mutuswami (1997).[8]

We also recall that a tree is an acyclic connected network, (a connected network for which there does not exist a sequence of nodes $i_0, i_1, \ldots, i_n$ such that $i_k i_{k+1} \in g$ for $i = 0, n-1$ and $i_0 = i_n$). Among trees, a particular network structure is the star.

Definition 4. A graph g is a star if there is some $i \in N$ such that $g = \{ik \mid k \in N, k \neq i\}$.

The distinguished individual i figuring in the definition will be referred to as the "hub."

3. Efficient and stable networks

In this Section, we characterize the set of efficient and stable networks, and provide some intuition for our results. The formal proofs are given in the Appendix. Our first Theorem characterizes efficient networks.

Theorem 1. *Suppose ϕ is a separable, convex function of individuals investments. Then, the unique efficient network is a star. Moreover, if ϕ is linear, then the unique efficient network is the symmetric star where the hub invests an equal amount in all links with peripheral agents.*

The intuitive explanation for this result is the following. The star is a minimally connected network, every peripheral agent concentrates his investment on a single link, and the distance between two nodes which are not directly connected is minimized. All these features contribute to making the star an obvious candidate for the efficient network. In fact, Bala and Goyal (2000) also show that the star is the unique efficient network architecture in the discrete link formation model. The novelty of Theorem 1 is that the set of networks on which we optimize is much richer than in Bala and Goyal (2000), as we allow agents to use weighted links. Hence, the main message of the Theorem is that *stars remain the unique efficient networks even if we consider a much larger set of weighted networks.* Because the set of networks on which we optimize is much larger, the proof of Theorem 1 is markedly different, and much more involved than the proof of Theorem 5.5 in Bala and Goyal (2000).

In the proof, we first show that, by reducing the number of links to form a star, the aggregate benefits of the network increase. By convexity, links become stronger, and the distance between nodes is reduced. However, the star that is formed is not necessarily feasible—it could involve the hub investing more than her endowment X. To solve this problem, we gradually reallocate the investments on the links in a way which increases the value of aggregate benefits. Finally, we show that the value of the network increases when we merge two stars into a single one.

Our second Theorem shows that the star is also the unique Nash stable network for strictly convex investments, and the only candidate for strongly pairwise stable networks with linear investments.

Theorem 2. (i) *If ϕ is strictly convex, the unique Nash stable network is a star where the hub invests all her endowment in a single link.*

(ii) *If ϕ is linear, a strongly pairwise stable network must be a star with $(n-1)$ peripheral nodes, and the set of strongly pairwise stable networks is nonempty iff $X \geqslant \frac{(n-1)^2}{n(n^2-3n+3)}$.*

Theorem 2 characterizes the set of Nash stable networks when ϕ is strictly convex, and strongly pairwise stable networks for linear ϕ. In both cases, stable networks are stars. With strictly increasing returns to scale, the hub invests in a single link to a peripheral agent; for constant returns to scale, the allocation of investment of the hub is indeterminate. The efficient symmetric star is stable as long as the endowment X is greater than $\frac{(n-1)^2}{n(n^2-3n+3)}$, an expression which is decreasing in n for $n \geqslant 3$, and converges to 0 as n goes to infinity.

[8] The concept of "strong pairwise stability" has been used by different authors under different names. See Gilles and Sarangi (2004) and Bloch and Jackson (2006) for an attempt to unify the terminology and a comparison of different stability concepts.

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In contrast to Bala and Goyal (2000), we characterize the star as the *unique* Nash stable architecture with a convex technology, and the *unique* candidate for strongly pairwise equilibrium with a linear technology. The strategy of the proof is related to Hojman and Szeidl's (2008) proof even though the models and the arguments are different. With a strictly convex link strength technology, we show that agents never have an incentive to invest in multiple links. Given this "one-link property," we progressively rule out all network architectures but the star.[9]

We first rule out cycles in equilibrium. Cycles arise when all agents invest fully on the link to their neighbor. Agents must then access the same indirect benefits through any node in the cycle, so that the cycle is fully symmetric. An agent then has an incentive to redirect his investment towards the neighbor who invests towards him, breaking the cycle towards a line but increasing the strength of his direct link.

Once cycles are ruled out, we show that the only candidate equilibrium among trees are stars. For any tree with diameter greater than or equal to three, we prove that there must exist terminal nodes who have an incentive to reallocate their investment in order to decrease the distance of their indirect connections. Finally, we provide an argument to show that disconnected stars cannot form in equilibrium.

When the link strength technology is linear, agents may invest in multiple links, and the marginal benefits of any connection must be equalized. We use this fact to construct joint pairwise deviations, and are able to show that, in any strongly pairwise equilibrium, if an agent invests in multiple links, all his neighbors must reciprocate by investing their entire endowment towards him.[10] This characterization enables us to use the same steps as in the case of strictly convex link strength technologies to rule out all network architectures but stars.

Taken together, Theorems 1 and 2 also help us understand the gap between efficiency and stability in networks with endogenous link strength. Efficient and stable networks are always stars, but the allocation of the hub's investment may differ. With increasing returns to investment, the hub invests all his investment in one link in equilibrium, but efficiency may require him to spread his investment across links, in order to increase the benefits of peripheral agents. When ϕ is linear, the symmetric star emerges as a strongly pairwise stable network when endowments are large enough. Since this is the unique efficient network, this result identifies a condition under which efficiency is compatible with stability when individual investments are perfect substitutes. It is interesting to note that the threshold value of X required to ensure this compatibility becomes smaller and smaller when the number of individuals becomes large.

4. Extensions

In this section, we discuss two extensions of the analysis, one dealing with an alternative model of investment, and the other with an alternative model of reliability.

4.1. Weakest link reliability

We now suppose that agents evaluate paths according to the value of the weakest link in the path rather than the product of link strengths. This notion of "weakest link reliability" is useful for physical communication networks, like the Internet, where the quality of a connection depends on the bottleneck of the network. Formally, we define the alternative notion of reliability as:

$$\hat{r}\big(p(i,j)\big) = \min_{s_{i^{m-1}i^m} \in p(i,j)} s_{i^{m-1}i^m}.$$

Agents choose to use the paths with the highest reliability, and we define the benefit of a connection from i to j in this setting as:

$$\widehat{R}(i,j) = \max_{p(i,j)\in P(i,j)} \min_{s_{i^{m-1}i^m} \in p(i,j)} s_{i^{m-1}i^m}.$$

With weakest link reliability, distance between nodes becomes irrelevant. Networks with different architectures but with identical distributions of link strengths result in the same value. Hence, efficient networks will typically not be

[9] This is also the structure of Hojman and Szeidl's (2008) proof, which first establishes that agents invest in a single link (Lemma 2), then shows that the distance between terminal nodes is at most two (Lemma 3) and finally rules out cycles and paths (Lemma 4).

[10] Notice that this argument exploits the fact that individual investment choices are continuous, and has no equivalent in discrete models of link formation.

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unique. The next theorem shows that any efficient network with weakest link reliability is equivalent to a star—the unique efficient network in our baseline model.

Theorem 3. *Let ϕ be a separable, convex function of individual investments, and suppose indirect benefits correspond to the notion of weakest link reliability. Then,*

(i) *Any efficient network is equivalent to a star.*

(ii) *If ϕ is linear, then any efficient network is equivalent to a symmetric star where the hub invests equal amounts on every link. Moreover, the symmetric star is strongly pairwise stable.*

(iii) *If ϕ is strictly convex, a star where the hub invests in a single link is strongly pairwise stable.*

The characterization of stars as efficient networks with weakest link reliability relies on arguments which are very similar to those showing that stars are optimal in our baseline model. The main difference between the two approaches is that efficient network architectures with weakest link reliability are not unique. In fact, when link strength technology is linear, any tree with links of equal strength is efficient, and it can be shown that links of equal strength can be obtained for any tree.[11] As stars are strongly pairwise stable, the set of strongly pairwise stable networks is nonempty.

A complete characterization of the set of stable networks is difficult. Since the value of a path depends on the weakest link along the *path*, a joint reallocation of investments by two players leading to more equal link strengths does not necessarily increase their payoffs. This lack of responsiveness of payoffs to individual choices prevents deviations and results in a large number of strongly pairwise stable networks. The following example illustrates why only a global reallocation of investments can improve individual payoffs.

Example 1. Let $\#N = 6$. Consider a line where $s_{i,i+1} > 0$ and $s_{ij} = 0$ if $j \neq i + 1$. In particular, let $s_{12} = s_{56} = 5/4X$, and $s_{23} = s_{34} = s_{45} = 7/6X$.

All individuals can gain if 2, 3, 4 and 5 relocate their investments so as to equalize link strength to $6/5X$. However, this joint deviation requires coordination among more than two agents and is not possible given our equilibrium concept. No pair of agent has an incentive to deviate from their strategies, so this inefficient network is strongly pairwise stable.

4.2. Investments as perfect complements

As a polar opposite to the case of perfect substitutes, we consider a model where investments are *perfect complements*, so that an agent only benefits from a relationship when the other agent also invests in the link. Formally,

$$s_{ij} = \min\left\{x_i^j, x_j^i\right\}.$$

When investments are perfect complements, agents should allocate "matching" investments on every link in order to maximize direct benefits. This intuition suggests that the efficient and stable network architectures will be very different from those obtained in the previous section. Stars will perform very badly, because the hub can only invest a small amount on every link. In contrast, regular networks where all agents invest the same amount on every link should perform fairly well.

Unfortunately, once indirect benefits are taken into account, the analysis of efficient and stable networks becomes intractable. We have to content ourselves with partial characterization results summarized in the following Theorem.

Theorem 4. *Suppose that individual investments are perfect complements. Then,*

(i) *An efficient graph cannot contain any component with three or more nodes which is a tree.*

(ii) *For $3 \leqslant n \leqslant 7$, the symmetric circle where every link has value $X/2$ is the unique efficient network.*

(iii) *Moreover, the symmetric circle is strongly pairwise stable.*

[11] To prove this last statement, one has to construct an algorithm, where terminal nodes invest their full endowment on their predecessor, who invest $X/(n-1)$ on their successors and $(n-2)X/(n-1)$ on their predecessors, etc.

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Theorem 4 establishes that trees cannot be efficient, and for small numbers of players (where distance does not matter too much), the unique regular network of degree 2 is the efficient network. However, for larger numbers of agents, efficiency may require a denser graph, where links are weaker, but distances between nodes shorter. Finally, we check that the set of strongly pairwise networks is nonempty and contains the symmetric circle.[12] Hence, comparing the case of perfect substitutes and perfect complements, we see that in the latter case, efficient and stable networks will be denser and more symmetric across players.

5. Conclusion

In this paper, we analyze the formation of communication networks when players choose how much to invest in each relationship. We suppose that players have a fixed endowment that they can allocate across links, and in the baseline model, suppose that link strength is an additively separable and convex function of individual investments, and that agents use the path which maximizes the product of link strengths. Under these assumptions, we characterize the optimal and stable networks. We also provide partial characterization results for alternative specifications of the investment technology and the benefit function.

In our view, this paper provides a first step in the study of networks where agents endogenously choose the quality of the links they form. One obvious drawback of our analysis is that agents are ex ante homogeneous. This assumption leads us to conclude that links will all be of the same quality (in the case of linear investments), or that, due to the hub's investments, some agents will be better connected than others (in the case of convex investments). Neither of these distributions of link intensities does justice to the broad array of social networks one observes in reality. In order to study the formation of networks with varying link quality, and the effect of individual characteristics on the efficient and stable distribution of link qualities, we need to introduce heterogeneity across agents. Following Rogers (2005), we could consider agents who differ both in their attractiveness (the intrinsic utility they bring to other agents), and their endowment. This seems to us to be a very promising avenue for future work.

Acknowledgments

We thank two anonymous referees and the associate editor for their perceptive comments. We have also benefited from discussions with C. Blackorby, S. Goyal and M. Jackson and from the comments of participants in seminars and conferences at Brescia, CORE, Guanajuato, University of Maryland at College Park, Montpellier, Montreal, Paris and Saint-Etienne.

Appendix A

Proof of Theorem 1. We prove the first statement in two steps. Consider any feasible component h of g of size m where the total amount of investment is mX.[13]

Step 1. We construct a feasible star S with higher aggregate utility than h, whenever h is not a star.

Step 2. If the graph g contains different components, we construct a single connected star which has higher aggregate utility than the sum of the stars.

Proof of Step 1. Order the link strengths of the component h so that:

$$z_1 \geqslant z_2 \geqslant \cdots \geqslant z_K.$$

Construct a star S by picking an agent at random (say agent m), and connecting him to the $(m-1)$ other agents with links of strengths $z_1, z_2, \ldots, \sum_{k=m-1}^{K} z_k$ in the following way:

[12] A complete characterization of the strongly pairwise stable networks is a difficult task. Since investments on a link are strategic complements, multiple equilibria typically arise, making the analysis of the entire game intractable.

[13] If the total amount invested in the component is strictly smaller than mX, then clearly the sum of utilities of agents in the component cannot be maximal.

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$$\text{for all } i = 1, \ldots, m-2, \quad x_i^m = \min\{\phi^{-1}(z_i), X\}, \qquad x_m^i = \phi^{-1}(z_i) - x_i^m,$$
$$x_{m-1}^m = \min\left\{\phi^{-1}\left(\sum_{k=m-1}^{K} z_k\right), X\right\}, \qquad x_m^i = \phi^{-1}\left(\sum_{k=m-1}^{K} z_k\right) - x_{m-1}^m.$$

Let $s_i, i = 1, \ldots, m-1$, denote the strengths of the $(m-1)$ links in the star. Notice that by construction,

$$s_i = z_i, \quad i = 1, \ldots, m-2, \quad \text{and} \quad s_{m-1} \geqslant z_{m-1}. \tag{1}$$

In this star, direct benefits are exactly equal to those of the component h. We show that indirect benefits have increased. In the star S, indirect benefits are given by:

$$I = 2 \sum_{i \neq j, i, j=1}^{m-1} s_i s_j.$$

Let $D = \{ij \mid i \text{ and } j \text{ are not neighbors in } h\}$. So, D is the set of pairs of nodes which are not directly connected in h, and so derive indirect benefits from each other.[14]

Suppose first that h is a *tree*, but not a star. For each pair i, j in D, let z_{t_i} and z_{t_j} denote the strengths of the two *terminal* links in the unique path $p^*(i, j)$. Clearly, for all $i, j \in D$,

$$R^p(i, j) \leqslant z_{t_i} z_{t_j}. \tag{2}$$

Moreover, since h is *not* a star, the (geodesic) distance between at least one pair of nodes, say i, j in h must be at least three. Hence, the inequality must be strict for such i, j since the maximum strength of any link is strictly less than one and the indirect benefit is the *product* of link strengths along the most reliable path. Also, note that each pair of nodes in D is associated with a *unique* pair of terminal links, and that one can construct exactly $\frac{(m-1)(m-2)}{2}$ pairs of terminal nodes out of the set $\{z_1, \ldots, z_{m-1}\}$. So, letting I' denote the sum of indirect benefits in h, the following inequality must be true:

$$I' < 2 \sum_{i \neq j, i, j=1}^{m-1} z_i z_j = I$$

where the last equality follows from Eq. (1).

Suppose now that h is not a tree, so that the cardinality of D is now strictly less than $\frac{(m-1)(m-2)}{2}$. We can again associate a *unique* pair of terminal links (z_{t_i}, z_{t_j}) to any pair of nodes $i, j \in D$.[15] Eq. (2) will hold again, aggregate indirect benefit in h is I', where

$$I' < 2 \sum_{i \neq j, i, j=1}^{m-1} z_i z_j \leqslant I. \tag{3}$$

The first inequality holds because there are now fewer than $\frac{(m-1)(m-2)}{2}$ pairs in D, and the sum is being taken over the product of the pairs which can be formed out of the strongest $(m-1)$ links.

If the star S is feasible, then this completes the proof. However, S may not be feasible. By construction, each peripheral agent invests no more than X. However, the construction may involve the center m investing more than X.[16]

We now show how to construct a *feasible* star from S which will have at least as large direct and indirect benefits as S, and hence strictly larger aggregate benefits than h.

Claim 1. *Consider the star S constructed above. If, for all* $i = 1, \ldots, m-1$, $x_i^m = X$, *then* $\sum_{i=1}^{m-1} x_m^i \leqslant X$, *and so the star S is feasible.*

[14] Note that if $ij \in h$, but the link strength is so weak that they do not derive direct benefits from each other, then h cannot be efficient; both i and j should switch their investment from ij to some other link.

[15] Since h is not a tree, there may be more than one most reliable path connecting i and j. The tie-breaking rule used to select some most reliable path—and hence the terminal links—is not important since the pair (z_{t_i}, z_{t_j}) can only be terminal links for the pair (i, j).

[16] However, the feasibility of h and convexity of ϕ ensures that the total resource used up in S does not exceed mX.

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Proof of Claim 1. Suppose all peripheral agents invest X on the link with m. Then,

$$\begin{aligned}\sum_{i=1}^{m-1} x_m^i &= \sum_{i=1}^{m-2} \phi^{-1}(z_i) + \phi^{-1}\left(\sum_{k=m-1}^{K} z_k\right) - (m-1)X \\ &\leqslant \sum_{i=1}^{K} \phi^{-1}(z_i) - (m-1)X \\ &\leqslant X.\end{aligned}$$

We thus observe that, if star S is not feasible, there must exist a link k such that $x_k^m < X$. Of course, $x_m^k = 0$.

Choose any i such that $x_m^i > 0$. Consider the new star $\bar{S}$ where the only change from S is that $\bar{x}_k^m = \min\{X, x_k^m + x_m^i\}$, and $\bar{x}_m^i = x_m^i - (\bar{x}_k^m - x_k^m)$. Let $\bar{s}_k - s_k = \epsilon$ and $s_i - \bar{s}_i = \delta$. Then, $\epsilon \geqslant \delta$ from convexity of ϕ. That is, this transfer of resource from link i to link k must (weakly) increase aggregate direct benefit.

We now check the effect of this change on indirect benefits. Let $\bar{I}$ and I be the aggregate indirect benefit in $\bar{S}$ and S respectively. Then,

$$\begin{aligned}\bar{I} - I &= 2\left[(\epsilon - \delta)\sum_{j\neq i,k} s_j + (s_k + \epsilon)(s_i - \delta) - s_k s_i\right] \\ &\geqslant 2(\epsilon s_i - \delta s_k - \epsilon\delta) \\ &\geqslant 2\epsilon(s_i - s_k - \delta) \\ &= 2\epsilon(\bar{s}_i - s_k) \\ &\geqslant 0.\end{aligned}$$

The last inequality holds because $\bar{s}_i = \phi(X) + \phi(\bar{x}_m^i) \geqslant \phi(X) \geqslant s_k$.

So, aggregate benefit is at least as high in $\bar{S}$ as it is in S. If $\bar{S}$ is not feasible, then we can continue to transfer resources from the hub to some peripheral node in the same way, until some star with aggregate benefit at least as high as S is feasible. □

Proof of Step 2. Consider two feasible stars S_1 and S_2 of sizes s_1 and s_2. Construct a new star $S^*$of size $s_1 + s_2$ centered around the hub of S, m_2, with the following investments:

$$\begin{aligned}x_i^{m_2 *} &= X \quad \text{for all } i \neq m_2, \\ x_{m_2}^{i*} &= x_{m_2}^i.\end{aligned}$$

In terms of direct benefits, the only change between the new star S^* and the stars S_1 and S_2 is that the hub of S_1 now invests fully on its link with m_2. Given the convexity of ϕ, this change must have weakly increased aggregate direct benefits. Consider next the indirect benefits in the new star, I^* and the sum of indirect benefits in the two stars, $I_1 + I_2$. Indirect benefits inside the star S_2 have not changed, and peripheral nodes of S_2 have gained access to new indirect connections. Agents in the star S_1 have gained access to new indirect connections to agents in star S_2. We focus attention on the difference in aggregate indirect benefits for agents inside the star S_1. New indirect connections linking the hub of S_1, m_1 to all the players in S_1 have been created. At the same time, the strength of an indirect connection between two peripheral nodes i and j of S_1 has been decreased from $(\phi(X) + \phi(x_{m_1}^i))(\phi(X) + \phi(x_{m_1}^j))$ to $\phi(X)^2$. The difference in indirect benefits can thus be computed as:

$$\begin{aligned}\Delta I_1 &= 2(s_1 - 1)\phi(X)^2 - \sum_{i,j\in S_1\setminus m_1} \phi(X)\left(\phi\left(x_{m_1}^i\right) + \phi\left(x_{m_1}^j\right)\right) - \sum_{i,j\in S_1\setminus m_1} \phi\left(x_{m_1}^i\right)\phi\left(x_{m_1}^j\right) \\ &= 2(s_1 - 1)\phi(X)^2 - 2(m-2)\phi(X)\sum_{i\in S_1\setminus m_1} \phi\left(x_{m_1}^i\right) - \sum_{i,j\in S_1\setminus m_1} \phi\left(x_{m_1}^i\right)\phi\left(x_{m_1}^j\right).\end{aligned}$$

By convexity of ϕ,

$$\sum_{i\in S_1\setminus m_1} \phi\left(x_i^{m_1}\right) \leqslant \phi(X)$$

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that is $W_i^j = W_i^k$. Hence, by reallocating investment to the link with j, agent i's utility cannot decrease. Consider now the utility to agent j of the connection to i:

$$W_j = s_{ij}\left(1 + s_{ik}W_j^{i,k} + \sum_{m\neq j,k} s_{im}W_j^{i,m}\right).$$

Hence, the reallocation of investment will result in a utility change

$$\Delta W_j = \varepsilon\left(1 + \sum_{m\neq j,k} s_{im}W_j^{i,m} + W_j^{i,k}(s_{ik} - s_{ij})\right) - \varepsilon^2 W_j^{i,k}.$$

For ε close to zero, the change in utility is positive as $s_{ij} \leqslant s_{ik}$.

Next, consider an agent l such that $x_j^l > 0$ and consider a reallocation where agent j shifts δ resources from the link to l to the link to i. Because x_j^i may be equal to zero, this reallocation may result in a utility loss for agent j. However, this utility loss must be continuous in δ and hence, for any ε, one can find $\delta(\varepsilon)$, such that the total effect of the reallocation of resources (δ, ε) on the utility of j is strictly positive. Now consider agent i. Since agent i could have chosen to connect directly to agent l, the equilibrium marginal value of i's connection to j must be at least as large as the marginal value of a possible direct connection to l. So,

$$W_i^j = 1 + s_{jl}W_i^{j,l} + \sum_{m\neq i,l} s_{jm}W_i^{j,m} \geqslant W_i^{j,l}. \tag{4}$$

Now the effect of the reallocation of resources δ on the utility of agent i is given by:

$$\Delta W_i = \delta\left(1 + \sum_{m\neq i,l} s_{jm}W_i^{j,m} + W_i^{j,l}(s_{jl} - s_{ij})\right) - \delta^2 W_i^{j,l}.$$

Using Eq. (4),

$$\Delta W_i \geqslant \delta(1 - s_{ij})W_i^{j,l} - \delta^2 W_i^{j,l},$$

which is strictly positive for δ close to zero. Hence, we have constructed a joint reallocation of resources which makes both players i and j strictly better off.

Case 2. Suppose $x_j^i = X$, but $x_k^i < X$.

From the proof above, it is clear that if $s_{ij} \geqslant s_{ik}$, then individuals i and k can jointly plan a profitable deviation. So, assume that $s_{ij} < s_{ik}$. Since $x_j^i = X$, this means that $X > x_k^i > 0$. Let $x_k^l > 0$ for some l. Since i invests on k, $W_i^k \geqslant W_i^l$. Similarly, since k invests on i, $W_k^i \geqslant W_k^j$. Suppose k transfers some resource from the link to l to the link to i. Then, i's total utility strictly increases since he gets the additional direct benefit, and there cannot be any loss in indirect benefit since $W_i^k \geqslant W_i^l$. Moreover, k is indifferent since $W_k^i = W_k^i$. For an exactly analogous reason, a transfer of resource by i from the link to j to the link to k makes k better off and leaves i indifferent. So, i and k have a profitable joint deviation. □

Lemma 3. *Let g be a connected network which is not a tree. Then,*

(i) *If ϕ is strictly convex, then g is not Nash stable.*

(ii) *If ϕ is linear, then g is not strongly pairwise stable.*

Proof. Let g contain a cycle $\{(12), (23), \ldots, (r-1, r), (r1)\}$. Denote $R = \{1, 2, \ldots, r\}$.

(i) Since ϕ is strictly convex, Lemma 1 implies that all agents invest only on one link. Let L_i and L_{i+1} be the sets of nodes contained in $N \setminus R$ which can be accessed through nodes $i, (i+1) \in R$. Note that these must be disjoint sets—since agents invest in only one link, there can be at most one cycle.

Let V_i be the total benefit that i gets from nodes in L_i. Similarly, let V_{i+1} be the total benefit that $i+1$ gets from L_{i+1}. We first show that $V_i = V_{i+1}$.

Suppose not. Let $V_i > V_{i+1}$. Then, $i-1$ is better off connecting to $i+1$ rather than to i. Hence, $V_i = V_{i+1} \equiv V$.

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So, i gets a benefit of $\phi(X)^k$ from node $i+k$ in R and $\phi(X)^k V$ from nodes in L_{i+k}. Let $B = 1 + V$. Then, if r is odd, agent i initially obtains

$$U_i = 2B \sum_{k=1}^{(r-1)/2} \phi(X)^k.$$

If r is even, then i obtains

$$V_i = 2B \sum_{k=1}^{(r-2)/2} \phi(X)^k + \phi(X)^{r/2} B.$$

Now suppose that agent i in the cycle unilaterally deviates and chooses to invest X on the link with $i-1$. Then, after the deviation, she receives

$$U_i' = 2B \sum_{k=1}^{(r-1)} \phi(X)^k.$$

Hence, irrespective of whether r is odd or even, this deviation makes agent i strictly better off.

(ii) We first prove that if ϕ is linear and g is strongly pairwise stable, then all agents in the cycle invest on only one link.

Suppose in contradiction that an agent i in the cycle invests on multiple links. From Lemma 2, both agents $i-1$ and $i+1$ must invest fully on the link with i. Let k be the smallest integer such that $i-k$ invests both on $i-k+1$ and $i-k-1$. Such k must exist to complete the cycle. Then, lemma 2 requires that $i-k+1$ invests fully on the link with $i-k$. But, by assumption $i-k+1$ invests fully on the link with $i-k+2$.

Now, take any agent j in L_i who is connected to i, and $k \in L_{i+1}$ who is connected to $i+1$. Lemma 2 implies that j invests fully on the link with i and k invests fully on the link with $i+1$. Repeated application of Lemma 2 implies that all agents in L_i and L_{i+1} in fact invest only on one link.

The rest of the proof is identical to that of (i) above.[17] □

Proof of Theorem 2. From Lemma 3, we know that only trees can be Nash stable if ϕ is strictly convex or strongly pairwise stable if ϕ is linear.

Now, consider a tree of diameter greater than or equal to 3. Let i and j be two terminal nodes at a distance greater than or equal to 3, and let k and l be their predecessors in the tree (where $k \neq l$ as the distance between i and j is greater than or equal to 3). Agent i could have chosen to invest his endowment on player l instead of k, so that

$$W_i^k \geqslant W_i^l = W_j^l + R(l,j) - R(l,i),$$

where the last equality results from the fact that by investing on l, agent i would have received the same marginal value as j, but would have in addition gained a connection of length 2 to j and lost the indirect connection between j and i. By a similar computation,

$$W_j^l \geqslant W_i^k + R(k,i) - R(k,j).$$

Summing up these inequalities, we obtain: $R(l,i) + R(k,j) \geqslant R(k,i) + R(l,j)$, a contradiction since $R(l,i) = R(k,i)R(k,l)$ and $R(k,j) = R(l,j)R(k,l)$ with $R(k,l) < 1$.

This shows that if g is not a collection of stars, then it cannot be Nash stable if ϕ is strictly convex or strongly pairwise stable if ϕ is linear.

Suppose that the network contains two stars, S_1 and S_2, with $|S_1| = s_1 \geqslant s_2 = |S_2|$. If ϕ is strictly convex, then from lemma 1, the hub of the star must invest fully on one link. Consider then a unilateral deviation from a peripheral agent i of S_2 in which the hub does not invest. By connecting to the hub of S_1, agent i obtains a payoff $V_i' = \phi(X) + (s_1 - 2)\phi(X)^2 + 2\phi(X)^2 = \phi(X) + s_1\phi(X)^2$ whereas she initially received a payoff of $V_i = \phi(X) + (s_2 - 3)\phi(X)^2 + 2\phi(X)^2 = \phi(X) + (s_2 - 1)\phi(X)^2$. Since $V_i < V_i'$, this shows that a network containing two or more

[17] Notice that in the proof of (i) we have used strict convexity of ϕ only to conclude that all agents invest only on one link.

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stars cannot be Nash stable. It is easy to check that a star in which the hub invests fully on one link is Nash stable. This concludes the proof of part (i) of the theorem.

If ϕ is linear, consider the peripheral agent i who receives the lowest benefit. Without loss of generality, suppose that she belongs to star S_2 and let i_2^* denote the hub of star S_2 and i_1^* the hub of star S_1. Consider the following joint deviation by agents i and i_1^*: agent i connects fully to $i_1^*, x_i^{i_1^*} = X$; agent i_1^* shifts away investment $x_{i_1^*}^j$ from some peripheral agent j in star S_1 and invests this amount in the link to i. After this deviation, agent i_1^*'s payoff has increased from $s_1 X$ to $(s_1 + 1)X$ and agent i's payoff has increased from V_i to $V_j + X s_{ii_1^*} > V_i$. This shows that if ϕ is linear, then a strongly pairwise network must be a connected star.

Suppose now that

$$X < \frac{(n-1)^2}{n(n^2-3n+3)}. \tag{5}$$

We want to show that no star can be strongly pairwise stable. Let g be any connected star with the hub being individual 1, and suppose that i and j are the two peripheral agents with lowest utility. This implies that $x_1^i {x_1}^j \leqslant (\frac{X}{n-1})^2$ since the hub's investment on i and j cannot exceed $2X/(n-1)$.

Consider a joint deviation in which players i and j invest fully on the link ij. Then, the gain in direct benefit for each of these two players is X. Each of these players loses an indirect benefit of $X(X + x_1^k)$ from each player $k \neq 1, i, j$ and the indirect benefit of $(X + x_1^i)(X + x_1^j)$. So, the total loss in indirect benefit to player l, $l = i, j$ is

$$\begin{aligned} L &= X \sum_{k \neq 1,i,j} \left(X + x_1^k\right) + \left(X + x_1^i\right)\left(X + x_1^j\right) \\ &= (n-1)^2 X^2 + x_1^i x_1^j \\ &\leqslant (n-1)^2 X^2 + \frac{X^2}{(n-1)^2} \\ &< X \end{aligned}$$

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To check this, note that the link s_{m-1} is the minimum in all comparisons, and so the sum of indirect benefits obtained between $m-1$ and other nodes is $2(m-2)s_{m-1}$. Similarly, s_{m-2} is the minimum in $(m-3)$ comparisons and so on.

Now, consider indirect benefits in h. Note that the arc z_{m-1} must also be involved in at least $(m-2)$ comparisons in the graph h (the minimum being attained if z_{m-1} connects some terminal node). Similarly, for any t, the t arcs with the lowest values, $\{z_{m-t}, \ldots, z_{m-1}\}$ must be the minimum in at least $(m-2)+(m-3)+\cdots+(m-t-1)$ connections (the minimum being attained if they both connect to some terminal node). This establishes that the sum of direct and indirect benefits is at least as high in S as in h.

Step 2. We now show that if the graph contains two stars S_1 and S_2, it is dominated by the graph where the two stars are merged into a single star, as in the proof of Theorem 1. By merging the two stars into a single star with hub m_2, direct benefits have increased. Furthermore, indirect benefits for players in star S_2 have strictly increased. For a peripheral agent i in star S_1, indirect benefits were equal to

$$\begin{aligned} I_i &= \sum_{j\in S_1\setminus m_1} \phi(X) + \min\{\phi(x^i_{m_1}), \phi(x^j_{m_1})\} \\ &= (m_1-1)\phi(X) + \sum_{j\in S_1\setminus m_1} \min\{\phi(x^i_{m_1}), \phi(x^j_{m_1})\}. \end{aligned}$$

In the new star, indirect benefits are given by:

$$I^*_i = (m_1+m_2-1)\phi(X).$$

As $m_2 \geqslant 1$ and $\phi(X) \geqslant \sum_{j\in S_1\setminus m_1} \min\{\phi(x^i_{m_1}), \phi(x^j_{m_1})\} I^*_i \geqslant I_i$ and indirect benefits cannot have decreased.

This concludes the proof of part (i).

(ii) Next suppose that ϕ is linear, and that the hub invests different amounts on its links with peripheral nodes. Let i be a node for which x^i_n is maximal, and j a node for which x^j_n is minimal. Consider a reallocation of investments, $\widetilde{x^i_n} = x^i_n - \varepsilon$, $\widetilde{x^j_n} = x^j_n + \varepsilon$. This reallocation does not affect direct benefits. For ε small enough, it only reduces indirect benefits between i and other players who are connected to the hub by links of maximal strength by ε. Indirect benefits between player j and all these players have been increased by ε, and indirect benefits between players i and j have strictly increased. Hence, this reallocation of investments has strictly increased the value of the graph, showing that the hub must put equal weight on all links with peripheral agents.

Now, in the symmetric star, each link has strength $(X + \frac{X}{n-1})$. Since there are $n-1$ links, each agent gets a benefit of nX. Since this equals the total resource availa(inw)-0.4(ing)-296.5(t)0.1(hat)]TJ0 -1.]TJ0 -1.]TJ0 -1.]TJ0 ect

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link is smaller than $\min\{X - x_k^i, X - x_l^j\}$, the investment in the direct link strictly increases the value of the graph, yielding a contradiction. This completes the proof of part (i).

(ii) To show that the symmetric circle is efficient for low values of n, consider the different values of n in turn. For $n = 3$, the circle is the only connected graph which is not a tree. Now, notice that direct benefits are equal to nX and hence are maximal in the circle. For $n = 4, 5$, we show that the circle also maximizes the value of indirect benefits. Notice first that the value of an indirect connection is always bounded above by $(X/2)^2$ as the middle player must allocate X over at least two links. For $n = 4$ and $n = 5$ all indirect connections in the circle are of length 2 and have value $(X/2)^2$. Hence, the circle achieves the highest sum of indirect links and is efficient. It is easy to check that any other allocation of investments results in a lower value of indirect links, so the circle with links of equal strength is uniquely efficient.

Suppose now that $n = 6, 7$. The indirect benefit for any node in the circle is

$$I = \frac{X^2}{2} + \frac{X^3}{4}.$$

Consider any other graph g. If this graph is to "dominate" the cycle, then at least one node (say i^*) has to derive an indirect benefit exceeding I. For each k, check that the circle maximizes indirect benefits from nodes at a distance of k. So, if i is to derive a larger indirect benefit in g, it must have more than two nodes at a distance of 2.[18]

It is tedious to show that the maximum indirect benefit that i^* can derive occurs when i^* has two neighbors, j_1, j_2, with each neighbor of i^* having three neighbors including i^* itself. Moreover, the optimum pattern of allocation from the point of view of i^* is

$$x_i^{j_1} = x_i^{j_2} = x_{j_2}^i = x_{j_2}^i = \frac{1}{2}.$$

This yields i^* a total indirect benefit of $\frac{X^2}{2} < I$. This completes the proof of part (ii).

(iii) Finally, we show that the symmetric circle is strongly pairwise stable. In the symmetric cycle, each i gets a direct benefit of X. No pattern of investment can result in higher direct benefits. So, we check whether a deviation by i and j can improve their indirect benefits.

Suppose i and j are neighbors in the cycle. Consider the effect on i of increasing investment to $\frac{X}{2} + y$ by both i and j on the link ij, and decreasing their investments on their other neighbors by y. The change in indirect benefit for i from j's other neighbor is $(\frac{X}{2} + y)(\frac{X}{2} - y) - (\frac{X}{2})^2 < 0$. A similar calculation shows that i also loses from nodes which are further away.

Suppose i and j are not neighbors in the cycle. Let i and j mutually invest y each on the link ij and simultaneously decrease investment on their previous neighbors by $\frac{y}{2}$. It is easy to check that this is the best possible deviation.

Clearly, this can only increase indirect benefit for i if there is some k such that the distance between i and k is now lower. This means that i accesses k through j. Let k be a neighbor of j. Then, the indirect benefit for i from k is

$$I = y\left(\frac{X - y}{2}\right) = \frac{Xy}{2} - \frac{y^2}{2}.$$

Now, i has reduced the strength of links with each of its previous neighbors by $\frac{y}{2}$. Also, since k is not at a distance of 2 from i in the cycle, there must be some node m, distinct from k which is at a distance of 2 from i. The loss in indirect benefit for i from m is

$$I' = \left(\frac{X - y}{2}\right)\frac{X}{2} - \frac{X^2}{4} = \frac{Xy}{2}.$$

Hence, the indirect benefit for i from k is lower than the loss in indirect benefit from m.

Repeating this argument, it can be shown that i's total indirect benefit will actually go down as a result of the deviation. □

[18] Since $n \leqslant 7$, the maximum distance between any two nodes in the circle is 3.

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Please cite this article in press as: F. Bloch, B. Dutta, Communication networks with endogenous link strength, Games Econ. Behav. (2008), doi:10.1016/j.geb.2008.03.007