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The Dynamic Structure of Optimal Debt Contracts*

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In a two-period model of costly state verification, the optimal contract is characterized and shown to exist. The optimal contract is interpreted as a bond contract. The model extends the result of one-period models that the verification region must be a left-tail interval to a multi-period setting. Conditions are identified for the optimal contract to exhibit features such as call (prepayment) option, coupon payment, or sinking fund. The optimality of certain bond covenants such as "refinancing covenants" and "dividends covenants" is also studied. *Journal of Economic Literature* Classification Numbers: 022, 026, 314, 521.

1. INTRODUCTION

The redemption provisions of most corporate bonds include coupon and/or sinking fund payment requirements and call (prepayment) options.[1] According to an extended version of the well-known Modigliani–Miller Theorem [8], in a perfect and frictionless economy, there is no reason why firms should systematically issue bonds that have these features. At a deeper level, one can ask why debt-looking financial contracts are systematically used at all.

The second question raised above has been answered by Townsend [14], Diamond [2], and Gale and Hellwig [3]. In each of these three papers, a debt-looking contract is derived as the optimal contract when lender can-

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[1] Coupon payments are interest payments made on a regular basis (usually semiannually) by a firm to its bondholders. Sinking fund provision typically requires the firm to repurchase

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not observe borrower's income without costs. Since their models are single period ones, the first question raised above cannot be addressed. The purpose of this paper is to consider a dynamic version of their models and to see whether the optimal contract will exhibit the features mentioned above.

We study a situation in which a firm (corporate insider) has an investment project. The project needs a one time investment at date zero and yields random cash flows at both date one and date two (revenues net of production cost, but gross of financial cost). The firm has no money, so the money needed is raised externally from outside investors through the capital markets. Given this environment, we analyze the optimal contract design problem faced by the firm.

One of the crucial assumptions of this paper is that without some costs being incurred, a contract cannot be made contingent upon the realized cash flows of the firm because it is costly to verify the true cash flows of the firm. An immediate consequence of this assumption is that if the payment to the outside investors varies with the reported cash flows, the firm may have incentives to misrepresent its cash flows. Since the firm has limited liability, the payment to the outside investors will have to be low when the cash flows are low. If there is no way to verify the true cash flows, the firm will always underreport its cash flows and the investors will not recover the capital they contributed.

One way by which this problem can be avoided is to have certain levels of reported cash flows verified. If a low cash flow report results in a verification, a firm with a high realized cash flow may not want to underreport. Of course, a verification is costly. Ex ante it is in the firm's interest to minimize the expected verification cost. Following Townsend [14] and Gale and Hellwig [3], we shall interpret a verification as bankruptcy. The verification cost is interpreted as bankruptcy cost, such as accounting cost, legal cost, or the cost of financial distress.

Although a contract cannot be made contingent on the true cash flows without incurring the verification cost, it can be made contingent on the reported cash flows. If a contract is structured such that the firm has no incentive to lie for any realized cash flows, it is called incentive compatible. By the Revelation Principle (see Green and Laffont [4], Harris and Townsend [5], or Myerson [10]), if an optimal contract exists, there is an incentive compatible contract that also achieves the optimum. In a dynamic setting, later period payments can depend on the current as well as the early period reports; this can alleviate the incentive problem.

Under the condition that date one verification cost is not decreasing in the firm's cash flow, we show that for an optimal contract, if a verification ever occurs at date one, it can occur only when the reported date one cash flow is below a critical level. This level, if positive, can be interpreted as a coupon payment or a sinking fund requirement. This result extends the result of one-period models that the verification region must be a left-tail interval to a multi-period setting.

The reason why coupons/sinking funds are used corresponds to the conventional wisdom that coupons or sinking funds provide regular tests of the firm's financial strength. It allows the investors to detect problems before it is too late and too costly to correct them.

If the verification cost at data two is increasing in the value of the firm's assets and if the firm is committed not to pay dividends at date one, an increase in the first period payment can reduce the retained earnings carried over to date two, hence reduce the expected bankruptcy cost. A way to achieve this is to give the firm an option to pay more when its date one cash flow is high. Of course, to present the firm from misrepresenting the true cash flow, the date two payment ought to be reduced accordingly. This feature corresponds to a call option in bond contracts.

Issues of bond covenants are also addressed in the paper. We show in the context of this simple model that it is optimal to have a convenant that prohibits any dividend payment at date one and a covenant that prohibits any subsequent borrowing at date one.

To improve our understanding of the financial contracts used in the real world, efforts have been made in recent years to incorporate market imperfections such as information and contracting costs into the theory of financial contracting. (See, for example, Jensen and Meckling [7], Myers [9], Ross [12], and Smith and Warner [13]). These studies provide us with important insights. In most of these studies, however, the forms of contracts are exogenously imposed (usually, debt and/or equity). Sometimes certain features, such as call or convertible options, are added on top of the basic contracts to see whether they have any welfare improving effect. No attempt has been made to derive the structure and the optimality of contracts from the basic attributes such as tastes, technologies, endowments, and information structure. In this paper the contractual form is derived instead.

In the next section, the model and the assumptions are presented. The constraints on the contracts are presented and the firm's optimization problem is formulated in Section 3. Section 4 characterizes the optimal contract. The optimal contracts is interpreted as a bond cotract in Section 5. The issues of bond covenants are treated in Section 6. Section 7 concludes the paper. All proofs are in the appendix.

2. The Model

A corporate insider (referred to as the insider or the firm hereafter) has a project, but has zero endowment. The project needs an investment of $1 at date zero and none afterward. It will yield cash flow Y_1 at date one and cash flow Y_2 at date two. Y_t is a random variable distributed on $[L_t, H_t)$ with distribution function F_t, where $H_t > L_t \geqslant 0$ for $t = 1, 2$. For simplicity, we assume that Y_1 and Y_2 are independent, $L_1 = L_2 = 0$, and $H_1 = H_2 = H$.

Since we shall interpret the optimal contract later as a debt contract, the model either applies to a privately-held firm or assumes away any conflict of interest between management and equityholders. Following is a crucial assumption:

A.1. A contract cannot be made contingent on the realized cash flow unless cash flow is verified ex post.

Verification is costly. Let $b_t(x_t)$ be the verification cost function at date [redacted]

A.2. b_1 and b_2 are non-decreasing and sufficiently smooth.

Both the insider and the investors are risk neutral and have time additive utility functions. The default-free interest rate per period is i. Without loss of generality, i is taken to be zero.

At date zero the insider designs a contract to maximize his expected utility subject to appropriate incentive constraints, the limited liability constraint, and the investors' individual rationality constraint (market constraint). We assume that the capital market is competitive and the investors earn the competitive rate. In addition, we shall temporarily make the following two assumptions.

A.3. The firm cannot borrow again after the realization of date one cash flow Y_1.

[redacted] ing. A.4 is a bond covenant that prohibits the payment of dividends at date one if the firm still has obligations outstanding at date two. Both A.3 and A.4 will be endogenized in Section 6. We will show there that in our model the firm will design a contract that has the properties of A.3 and A.4. The next two assumptions are technical.

A.5. F_t is continuously differentiable with strictly positive density function f_t, where $t = 1, 2$.

A.6. The hazard rate for Y_2, $f_2/[1 - F_2]$ is increasing.[2]

[2] Most commonly encountered distributions have this properties.

Although the payments to the investors cannot be made contingent on the true cash flows unless a verification occurs, the payments and verification can depend on the firm's reported cash flows. These payments, however, should be arranged such that the firm does not have incentives to lie. That is, they should be incentive compatible. By the Revelation Principle, there is no loss of generality in considering only the incentive compatible contracts in solving a contract design problem. (See Myerson [10], Harris and Townsend [5], and Green and Laffont [4]).

A contract is a pair of payment schedules and a pair of verification schedules (one for each date). Let $D_1(y_1)$ be the date one verification schedule when y_1 is reported, and let $D_2(y_1, y_2)$ be the date two verification schedule when y_1 and y_2 are reported. D_t is the probability that a verification will take place. In this paper, we shall not consider randomized verification schedules. That is, D_t is either 0 or 1. We also assume that the investors can commit to the verification schedules D_1 and D_2 ex ante. This may be rationalized by the effects of reputation.

Let $P_1(y_1)$ be the payment to the investors at date one after y_1 is reported and $D_1(y_1)=0$. Let $P_2(y_1, y_2)$ be the payment at date two after y_1 and y_2 are reported and $D_1(y_1)=0$. So P_1 and P_2 are the payment schedules when no verification occurs at data one. Let p_1 and p_2 be the corresponding payment schedules when a verification occurs at date one. Note that p_t and P_t are defined as the payment to the investors gross of verification cost.[3]

The order in which the events take place is as follows: at date zero, the contract is signed and \$1 is invested; at date one, Y_1 is realized and the firm reports y_1; based on y_1, verification will be made with probability $D_1(y_1)$ and the payment $P_1(y_1)$ or $p_1(y_1)$ will be made to the investors according to the contract; at date two the same happens.

3. The Formulation of the Firm's Problem

In this section, we first apply the result of the one-period problem to the last period. That allows us to simplify the notations. Then we derive the firm's limited liability (wealth) constraints and incentive compatible constraints at data two, and the investors' individual rationality constraint. The firm's optimization problem is then formulated.

Since date two is the last date, for a given y_1 the results of Townsend [14] or Gale and Hellwig [3] apply. If a contract is optimal, then its payment schedule at date two, $P_2(y_1, y_2)$, must be independent of y_2 when no

[3] I could use the same notations for payment schedules following both $D_1=0$ and $D_1=1$. Th[illegible] that I use different notations for the payment schedules depending on whether a [illegible]

verification occurs at date two. Denote this payment by $R_2(y_1)$ (or $r_2(y_1)$ when verification occurred at date one). Note that $R_2(y_1)$ depends on y_1 alone whereas $P_2(y_1, y_2)$ depends on both y_1 and y_2. We will refer to $R_2(y_1)$ (or $r_2(y_1)$) as the required payment at date two.

A verification occurs at date two if and only if the reported date two cash flow plus the retained earnings fall short of this required payment. Furthermore since both parties to the contract are risk neutral, the optimal contract is the one that minimizes the expected verification cost. By increasing the amount paid to the investors in the event of verification, the contract can reduce the region over which verifications need to take place, hence reduce the expected verification cost.

PROPOSITION 1. *For an optimal contract, $D_2(y_1, y_2)=1$ if and only if $[y_1 - P_1(y_1)] + y_2 < R_2(y_1)$. If a verification does not occur at date two, $P_2(y_1, y_2) = R_2(y_1)$, which is independent of the reported cash flow at date two. In the event of verification, all the assets the firm possesses are paid to the investors. That is, $P_2(y_1, y_2) = [y_1 - P_1(y_1)] + y_2$. The same results hold when P_t and R_2 are replaced with p_t and r_2.*

By this proposition, at date two, the firm pays all it has to the investors when a verification occurs; it pays $R_2(y_1)$ (or r_2) otherwise. So once the [illegible] teristics of the optimal contract at date two are determined.

Now let us see what constraints on $D_1(y_1)$, $p_1(y_1)$, $r_2(y_1)$, $P_1(y_1)$, and $R_2(y_1)$ are needed so that they provide the correct incentives for the firm to report y_1 truthfully. By A.3, no borrowing is possible after date zero; the date one payment P_1 cannot exceed the true realized cash flow y_1. This gives us the limited liability constraints

$$P_1(y_1) \leqslant y_1, \qquad p_1(y_1) \leqslant y_1. \tag{1}$$

Suppose x_1 and y_1 are realizations of Y_1 for which a date one verification does not occur, i.e., $D_1(x_1) = D_1(y_1) = 0$. Suppose it is feasible for the firm with y_1 to mimic that with x_1, i.e., $y_1 \geqslant P_1(x_1)$. Given that the date two required payment depends on y_1 only and given that no dividends can be paid at date one, the firm can have positive income if and only if y_2 is above the *date two net liability*, $R_2(y_1)$ minus the retained earnings $[y_1 - P_1(y_1)]$. The incentive compatible constraint that the firm with y_1 does not want to report x_1 is

$$\int_0^H \operatorname{Max}\{y_2 - [R_2(y_1) - y_1 + P_1(y_1)], 0\}\, dF_2$$
$$\geqslant \int_0^H \operatorname{Max}\{y_2 - [R_2(x_1) - y_1 + P_1(x_1)], 0\}\, dF_2$$

which is equivalent to

$$P_1(y_1)+R_2(y_1)\leqslant P_1(x_1)+R_2(x_1) \tag{2}$$

for y_1 and x_1 such that $y_1 \geqslant P_1(x_1)$. We shall call P_1+R_2 the firm's *total liability.* Since for $y_1 \geqslant x_1$, $y_1 \geqslant x_1 \geqslant P_1(x_1)$, (2) implies that the total liability cannot increase in the non-verified region.

Suppose $D_1(y_1)=1$ and $D_1(x_1)=0$ and suppose $y_1 \geqslant P_1(x_1)$. The incentive constraint that the firm with y_1 will not report x_1 is

$$\int_0^H \text{Max}\{y_2-[r_2(y_1)-y_1+p_1(y_1)],0\}\,dF_2$$

$$\geqslant \int_0^H \text{Max}\{y_2-[R_2(x_1)-y_1+P_1(x_1)],0\}\,dF_2$$

which implies

$$p_1(y_1)+r_2(y_1)\leqslant P_1(x_1)+R_2(x_2). \tag{3}$$

If $D_1(x_1)=1$, then the firm with y_1 will not misrepresent itself because the lie will be discovered and severe punishment can be imposed. So (2) and (3) are all the incentive compatible constraints we need.

For a given y_1 for which $D_1(y_1)=1$, the expected payoff to the investors is $p_1(y_1)-b_1(y_1)+\int_0^H p_2(y_1, y_2)\,dF_2(y_2)$; for y_1 for which $D_1(y_1)=0$, the expected payoff is $P_1(y_1)+\int_0^H P_2(y_1, y_2)\,dF_2(y_2)$, where $p_2(y_1, y_2)$ and $P_2(y_1, y_2)$ are defined in Proposition 1. The investors' individual rationality constraint is

$$\int_0^H D_1(y_1)\{p_1(y_1)-b_1(y_1)+\int_0^H p_2(y_1, y_2)\,dF_2(y_2)\}\,dF_1$$

$$+\int_0^H [1-D_1(y_1)]\{P_1(y_1)+\int_0^H P_2(y_1, y_2)\,dF_2(y_2)\}\,dF_1 \geqslant 1. \tag{4}$$

Since everyone is risk neutral, the problem of maximizing the insider's expected payoff subject to the constraints (1), (2), (3), and (4) is equivalent to the problem of minimizing the expected verification cost subject to the same constraints, by choosing D_1, p_1, r_2, P_1, and R_2.

$$\text{Min} \int_0^H D_1(y_1)\left\{b_1(y_1)+\int_0^{r_2-(y_1-p_1)} b_2(y_2+(y_1-p_1))\,dF_2\right\}dF_1$$

$$+\int_0^H [1-D_1(y_1)]\int_0^{R_2-(y_1-P_1)} b_2(y_2+y_1-P_1)\,dF_2\,dF_1$$

$$\text{s.t. (1), (2), (3), (4),}\quad \text{and}\quad D_1, D_2=0 \text{ or } 1. \tag{5}$$

4. The Characterization of the Optimal Contract

The existence of an optimal solution to (5) is proved later. The optimal contract is characterized through the propositions presented below.

The incentive compatible constraint (3) says that the firm with a y_1 that is in the verified region does not want to claim to be in the non-verified region even if it is feasible to do so ($y_1 \geqslant P_1(x_1)$). The proposition below shows that (3) cannot be binding for the optimal contract. So it can be dropped.

PROPOSITION 2. *For the optimal contract,* (3) *cannot be binding. That is, the insider is strictly better off in the verified region than in the non-verified region.*

The idea behind the proof of Proposition 2 is this: if, for the optimal contract, the firm is not strictly better off in a verified state y_1 than in a non-verified state x_1 when a lie is possible ($y_1 \geqslant P_1(x_1)$), we can redefine y_1 as a non-verified state and define $P_1(y_1)+R_2(y_1)=p_1(y_1)+r_2(y_1)=P_1(x_1)+R_2(x_1)$ with $P_1(y_1)=y_1$. The contract is otherwise unchanged. Since the firm's total liability at y_1 is unchanged, it will report y_1 truthfully as before. In addition, no other type will mimic y_1 because, otherwise, this would mean that in the original contract some type would mimic x_1. Now in the redefined contract, the investors will receive a high payoff because the verification cost $b_1(y_1)$ is saved. So this redefined contract dominates the old contract. The old contract cannot be optimal.

PROPOSITION 3. *If $b_2(\ )$ is strictly increasing, then $P_1(y_1)=y_1$ for y_1 satisfying $y_1-P_1(y_1)<R_2(y_1)$.*

The intuition is simple: if b_2 is strictly increasing and if a part of date one cash flow is retained, this will increase the amount of assets being verified when a verification occurs at date two, hence increase the expected verification cost. By increasing P_1 and simultaneously reducing R_2 so that their sum does not change, one can reduce the expected verification cost and increase the investor's payoff without changing the firm's payoff.

When y_1 is above $P_1(y_1)+R_2(y_1)$, y_1 is enough to retire the firm's total [redacted] case.

Proposition 4 below characterizes the payment schedules p_1 and r_2 in the region where a verification occurs at date one.

PROPOSITION 4. *For the optimal contract,* $p_1(y_1)+r_2(y_1)-y_1$ *is a* *constant k. Furthermore, if $b_2(\,)$ is strictly increasing, $p_1(y_1)=y_1$ and $r_2(y_1)=k$.*

In the region where the cash flow is not verified at date one, by (2) the firm's date two net liability $R_2-(y_1-P_1)$ has to decrease with reported y_1 to provide the firm with incentives to report truthfully. But in the region where the date one cash flow is verified, private information is revealed, so there is no need to provide the firm with incentives to report y_1 truthfully. The date two net liability $r_2-(y_1-p_1)$ should be set independent of y_1 at the "first best" level k, the level making the benefit of increasing the date two required payment equal to the expected cost of such an increase. When b_2 is increasing, for the same reason as in Proposition 3, the firm should pay all it has at date one.[4]

The next proposition shows that in the region where reports are not verified ($D_1=0$), the incentive compatibility constraint (2) must be binding for the optimal contract.

PROPOSITION 5. *For the optimal contract, $P_1(y_1)+R_2(y_1)=M$ for all y_1 for which $D_1(y_1)=0$. That is, the incentive compatible constraint* (2) *is binding.*

This result is similar to a result in most other models of asymmetric information. If the incentive compatible constraint is not binding at the optimum, then the contract can be improved by moving it towards the first best. More specifically, we show that if (2) is not binding (if the total liability is not a constant), then the date two net liability $P_1+R_2-y_1$ must be at the "first best" level k as in the case in which y_1 is verified. But this implies that the total liability is increasing ($P_1'+R_2'=1$) and (2) is violated.

By Propositions 4 and 5, the total liability is a constant M in the non-verified region and the date two net liability is a constant k in the verified region.

PROPOSITION 6. *For the optimal contract, a verification occurs (if it occurs at all) at date one if and only if the reported y_1 is below some critical level m. Formally, $D_1(y_1)=0$ if and only if $y_1\leqslant m$, where m is in $[0, H]$.*

The proof of this result follows this reasoning: Suppose at v, $D_1(v)=0$. That is, suppose it is better to make v a non-verified state than to make it a verified state. Then we can show that for y_1 sufficiently close to and above v, the benefits of making y_1 a non-verified state are always greater than the benefits of making it a verified state. So y_1 should be in the non-

[4] Of course, this result depends on the assumption that Y_1 and Y_2 are independent. If they are positively correlated, $r_2(y_1)$ will generally be increasing in y_1 because higher y_1 means higher Y_2. The optimal $r_2(y_1)$ will be higher.

verified region for an optimal contract. Following this reasoning, all y_1 above v must be a non-verified state. This rules out the possibility that a non-verified region lies below a verified region.

Note that Proposition 6 is an implication of optimality and incentive [redacted] and b_2 are equal to a same constant; then it can be shown that a verification need not occur at date one.

However, when $b_1(y_1)$ or $b_2(y_1)$ is increasing in y_1, a verification can occur at date one. This is because if no verification occurs at date one, R_2 may have to be set at a very high level in order to punish a low reported y_1. This will increase the probability as well as the expected cost of verification at date two. So there is a trade-off between verifying at date one and verifying at date two. If the cost of verification at date one is not too large for low y_1, then it will pay to set D_1 to 1 and reduce the date two required payment for low reported y_1.

In practice, the conventional wisdom is that coupons or sinking funds provide regular tests of the firm's financial strength. It allows the investors [redacted] paragraph.

Now we shall show that the above characterization is not vacuous; an optimal contract does exist. We use Cesari's existence theorem [1] on control problems. In fact, Cesari's theorem applies to control problems with bounded constraints. Our problem is unbounded. Some adjustments are made to apply it here.

PROPOSITION 7. *There exists an optimal solution to* (5).

5. INTERPRETATION OF THE OPTIMAL CONTRACT AS A BOND CONTRACT

As in Townsend [14] and Gale and Hellwig [3], a verification is interpreted as bankruptcy. Although this interpretation is not perfect, it captures some realism and it offers a convenient way to model financial contracts. If we take b_t as all the costs of enforcing a contingency, the interpretation can be more general. The verification cost is just a part of the enforcements costs.

From the last section, we know that in the optimal contract, when $y_1 \geqslant m$, verification will not occur at date one and the sum of the firm's date one payment P_1 and its date two required payment R_2 is a constant

M. However, if b_2 is not strictly increasing, the division of M between P_1 and R_2 is undetermined except that P_1 must satisfy the constraint (1) $P_1 \leqslant y_1$. For example, both $P_1 = m$, $R_2 = M - m$ and $P_1 = y_1$, $R_2 = M - y_1$ are possible optimal schedules.

If b_2 is strictly increasing, $P_1 = y_1$ for y_1 in $[m, M]$ by Proposition 3. It is important to note that the schedule $P_1 = y_1$ for $M > y_1 \geqslant m$ is not a [illegible] always report m, hence only pay m at date one and retain $y_1 - m$. Therefore, we interpret the contract as containing a call provision .

The geometrical representation of the optimal contract derived in last section is shown in Fig. 1. In Fig. 1, we assumed that when y_1 is above M, P_1 stays at M and R_2 stays at 0. However, any division of M between P_1 and R_2 is optimal. Note that by Proposition 2, $M - m$ is higher than k so

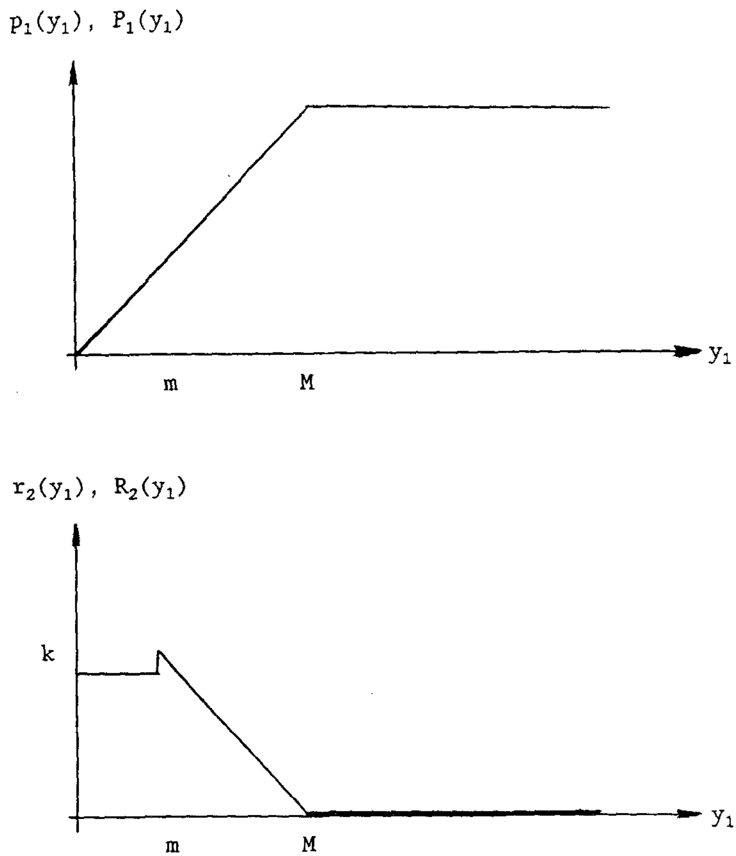

FIG. 1. The optimal payment schedules p_1 and r_2 (for $y_1 < m$), P_1 and R_2 (for $y_1 \geqslant m$) as functions of the reported y_1.

there is a jump at $y_1 = m$ in the lower schedule. The analytic expressions for the schedules of the optimal contract are tabulated below:

y_1	D_1	R_1	R_2	r_1	r_2
$y_1 < m$	1	/	/	y_1	k
$m \leqslant y_1 < M$	0	y_1	$M - y_1$	/	/
$y_1 > M$	0	M	0	/	/

In this optimal contract, the firm generally is required to pay a minimum amount m at date one and it has an option to pay up to M. When it chooses to pay more at date one, the date two payment reduces accordingly. This resembles a bond contract with a coupon (or sinking funds) and a call option. It can be shown that the optimal contract can be written in terms of the characteristics of a bond contract such as face value, coupon rate, call price, and sinking fund requirement. So the optimal contract can be called a bond contract.

6. Bond Covenants

The optimal contract is characterized under the assumptions that the firm cannot borrow after the realization of Y_1 and that it cannot distribute dividends at date one (A.3 and A.4). Bond covenants are studied in Smith and Warner [13]. Two of the most common covenants are referred to as "dividend covenants" and "covenants restricting subsequent financing policy." We shall call the latter refinancing covenants for simplicity. A.3 represents the most restrictive refinancing covenant and A.4 represents the most restrictive dividend covenant.

In this section, we show that it is in the firm's interest to include these two covenants in the bond contract. Of course, the fulfillment of the dividend covenant depends on the lenders' ability to observe the firm's dividend distributions. In many case, it is costly for the lenders to do this. In this section, we also explore the effects on the optimal contract when the lenders are unable to make the firm to retain any earnings.

First let us note that when the firm is allowed to borrow from a third [redacted] old and the new lenders' contracts and view the combined contract as being issued to the old lenders. This composite contract must be incentive compatible and individually rational for the lenders because the two components of it are. So the composite contract must be in the constrained (feasible) set of problem (5) in Section 3, As a result, it cannot be better than the optimal contract derived there.

Furthermore, introducing new lenders may further constrain the feasible set. This is because more incentive compatibility constraints are needed when the opportunity of reborrowing is introduced.[5] We shall not analyze these constraints in this paper due to their complexity. It suffices to say that the optimal contract derived will be at least as good.

If the insider is allowed to pay himself some dividends out of the firm's date one cash flow, he will certainly do so because he may not get them if they are retained. When dividends are paid at date one, the probability of a verification and expected costs of verification will rise. Since the insider bears the cost of verification, he will be worse off.

As an extreme example, suppose the firm can pay as much dividends as it wants out of the date one cash flow. Then at date one, the firm will pay the lenders only m, the level just high enough to avoid a verification, and pay the shareholder $y_1 - m$ as dividends. Anticipating this, the lenders will require a higher R_2 (or M) to protect themselves. There may even be cases in which m also has to be raised. This will certainly increase the probability as well as the expected costs of bankruptcy. Hence, the insider is worse off ex ante.

This example raises an interesting question: what is the optimal contract if the firm can pay as much dividends as it wants out of the date one cash flow? In this case, anything not paid to the lenders will be paid to the insider at date one. The incentive compatibility in the non-verified region will be

$$y_1 - P_1(y_1) + \int_{R_2(y_1)}^{H} [y_2 - R_2(y_1)]\, dF_2$$

$$\geqslant y_1 - P_1(x_1) + \int_{R_2(x_1)}^{H} [y_2 - R_2(x_1)]\, dF_2.$$

The differential version of this will be $[1 - F_2(R_2)]\, P_1' + R_2' \leqslant 0$.

It can be shown that in this case, results similar to Propositions 3, 4, 5, and 6 will hold. In the absence of dividend covenant, the minimum date

[5] This idea is related to the results of a number of papers on the theory of equilibrium with moral hazard (see for example, Pauly [11] and Helpman and Laffont [6]). They show that trading with third parties will affect the welfare of the original contract parties. Feasible allocations will be affected. Additional incentive compatible constraints will be needed. Therefore, there is a value to exclusion of third parties. I thank a referee for pointing this out to me.

one payment m will have a new function: it forces the firm to pay at least m at date one. For example, when dividend payment can be prohibited and when the verification costs are equal to a constant, the firm is indifferent between having required payment 0 at date one and M at date two and having m at date one and $M-m$ at date two. But in the absence of dividend covenant, the firm will prefer the latter.

The optimal contract without dividend covenant will be the same as that shown in Fig. 1 except that the declining portion of R_2 is a convex curve with a slope of $1/[1-R_2]$. The call price will be a decreasing function of y_1. That is, each additional y_1 will retire more date two obligation R_2. In contrast, for the contract in Fig. 1, the call price is a constant, 1. In general, the call prices we observe in practice are constants.

Of course, reality is somewhere between these two extreme cases. The borrowers will be able to hide cash or non-cash distributions to themselves to a certain extent.

7. CONCLUSION

In a simple two-period model, this paper characterizes the structure of the optimal contract when it is costly to verify the firm's true cash flows. It extends the result from one-period models that the verification region must be a left-tail interval to a multi-period setting. The optimal contract exhibits some features we commonly observe in corporate bond contracts. The model is theoretically more appealing because the contractual form is not assumed but derived from first principles.

To my knowledge, no theoretical work has explained why coupon/sinking fund payments are widely used. In finance literature a popular rationale for call provisions is that they provide a way to overcome agency costs caused by the existing bonds. When an agency problem arises, the firm can avoid it by calling the bonds. An implicit assumption in this type of analysis is that after the bonds are issued, the firm will have enough money in the future to call them back. Where the money comes from is not specified. If the money comes from the firm's other assets, then by the firm's using the money as collateral for the bonds or restricting the payment of it to the shareholders, the bonds will be default risk free and the agency problem will not be there in the first place. On the other hand, if the bonds are retired by the firm's issuing new securities, the agency costs associated with issuing new securities will have to be specified.

The model presented here is still a simple one. It cannot differentiate between coupon and sinking funds. The firm's reinvestment opportunities are ignored. These topics are left to future research.

APPENDIX

Proof of Proposition 1. See Gale and Hellwig [3].

Proof of Proposition 2. Suppose (3) is binding for a pair of eligible y_1 and x_1. That is, there exist y_1 and x_1 such that $D_1(y_1)=1$, $D_1(x_1)=0$, $y_1 \geqslant P_1(x_1)$), and

$$p_1(y_1)+r_2(y_1)=P_1(x_1)+R_2(x_1).$$

Then we can show that there exists a feasible contract which dominates the purported optimal contract. Let us redefine the contract at y_1 so that y_1 becomes a non-verification state and redefine $P_1(y_1)$ and $R_2(y_1)$ so that $P_1(y_1)+R_2(y_1)=p_1(y_1)+r_2(y_1)$ with $P_1(y_1)=y_1$. Now under the redefined contract, the investors get more at y_1 because they do not have to pay the verification cost. The insider's expected payoff at y_1 is unchanged, so the insider with y_1 will not mimic other types because he does not want to mimic under the old contract. Hence, the redefined contract must be incentive compatible if we can show that no other types will want to mimic y_1. Suppose there exists a y_1' such that $y_1' > P_1(y_1)=y_1$ and $P_1(y_1)+R_2(y_1)<P_1(y_1')+R_2(y_1')$. Then since $y_1' > y_1 \geqslant P_1(x_1)$, we have $P_1(y_1')+R_2(y_1')>P_1(x_1)+R_2(x_1)$ because $P_1(x_1)+R_2(x_1)=P_1(y_1)+R_2(y_1)$. That is, y_1' must prefer reporting x_1 under the old contract. So the original contract could not be incentive compatible. This is a contradiction. Therefore, the redefined contract dominates the old contract because it saves the verification cost $b_1(y_1)$; the old contract cannot be optimal as supposed. Q.E.D.

Propositions 4, 5 and 6 are proved by transforming (5) into a control problem (proof of Proposition 3 is trivial once proposition 4 is proved). We need a differential version of (2):

$$P_1' + R_2' \leqslant 0. \tag{a.1}$$

[redacted] more restrictive (2).

Now with the control variables u_1 and u_2 and with the constraint (a.1), (5) becomes an optimal control problem. We are interested in characterizing P_1 and R_2 in this proof. Let m_1 and m_2 be the costate variables for P_1 and R_2, respectively. Let λ and T be the multipliers for (4) and (a.1), respectively. Let Q and q be the multipliers for the two constraints in (1), respectively. The Lagrangian (Hamiltonian) for the problem is

$$
\begin{aligned}
L = D_1 \Bigg\{ & b_1 - \lambda(p_1 - \lambda(p_1 - b_1) + \int_0^{r_2 - y_1 + p_1} b_2(y_2 + y_1 - p_1)\, dF_2 \\
& - \lambda \Bigg[\int_0^{r_2 - y_1 + p_1} [y_2 + y_1 - p_1 - b_2(y_2 + y_1 - p_1)]\, dF_2 \\
& + r_2[1 - F_2(r_2 - y_1 + p_1)] \Bigg] + q(p_1 - y_1) \Bigg\} f_1 \\
& + (1 - D_1) \Bigg\{ \Bigg\{ - \lambda P_1 + \int_0^{R_2 - y_1 + P_1} b_2(y_2 + y_1 - P_1)\, dF_2 \\
& - \lambda \Bigg[\int_0^{R_2 - y_1 + P_1} [y_2 + y_1 - P_1 - b_2(y_2 + y_1 - P_1)]\, dF_2 \\
& + R_2[1 - F_2(R_2 - y_1 + P_1)] \Bigg] \\
& + T(u_1 + u_2) + Q(P_1 - y_1) \Bigg\} f_1 + [m_1(u_1 - r_2') + m_2(u_2 - R_2')] \Bigg\}.
\end{aligned} \tag{a.2}
$$

By Proposition 2, the constraint (3) has been ignored.

Proof of Proposition 4. Consider the region for which $D_1 = 1$. From (a.2), the first-order conditions (after simplifications) for p_1 and r_2 are

$$
\begin{aligned}
\Bigg\{ & -(1 + \lambda) \int_0^{r_2 - y_1 + p_1} b_2'(y_2 + y_1 - p_1)\, dF_2 \\
& + q + (1 + \lambda)\, b_2(r_2)\, f_2(r_2 - y_1 + p_1)) \\
& - \lambda[1 - F_2(r_2 - y_1 + p_1))\Bigg\} f_1 = 0,
\end{aligned} \tag{a.3}
$$

$$
\{(1 + \lambda)\, b_2(r_2)\, f_2(r_2 - y_1 + p_1)) - \lambda[1 - F_2(r_2 - y_1 + p_1))\} f_2 = 0. \tag{a.4}
$$

These two equations imply

$$
(1 + \lambda) \int_0^{r_2 - y_1 + p_1} b_2'(y_2 + y_1 - p_1)\, dF_2 = q(y_1). \tag{a.5}
$$

Suppose that $G(y_1) = p_1(y_1) + r_2(y_1) - y_1$ is not a constant. Then there exists an interval (x, y) over which $G(y_1)$ is strictly monotone. (a.4) implies $(1 + \lambda)\, b_2(r_2) = \lambda[1 - F_2(G(y_1))]/f_2(G(y_1))$. By the assumption of increasing hazard rate (i.e., decreasing $[1 - F]/f$), $b_2(r_2(y_1))$ must be strictly monotone in (x, y). This implies that $b_2'(r_2(y_1))$ is positive in (x, y) (note that b_2' is non-decreasing). So $q(y_1)$ in (a.5) must be positive in (x, y)

because when y_2 approaches the upper limit $r_2 - y_1 + p_1$ of the integral, $b_2'(\ )$ approaches $b_2'(r_2(y_1))$ which is positive in (x, y). Since q is the multi-plier for constraint $y_1 \geqslant p_1$, a positive q means $p_1 = y_1$ in (x, y). Substituting this back into (a.4), we have $(1+\lambda)\, b_2(r_2) = \lambda[1 - F_2(r_2))]/f_2(r_2))$. This implies that r_2 must be a constant k in (x, y). Hence $p_1(y_1) + r_2(y_1) - y_1 = r_2$ must be a constant in (x, y). But this is contradiction. Q.E.D.

Proof of Proposition 5. The first order conditions with respect to u_1, u_2, P_1, and R_2 are

$$m_1 + T = 0, \tag{a.6}$$

$$m_2 + T = 0, \tag{a.7}$$

$$\begin{aligned} -m_1' = \Big\{ &-(1+\lambda) \int_0^{R_2 - y_1 + P_1} b_2'(y_2 + y_1 - P_1)\, dF_2 \\ &+ Q + (1+\lambda)\, b_2(R_2)\, f_2(R_2 - y_1 + P_1)) \\ &- \lambda[1 - F_2(R_2 - y_1 + P_1))\Big\} f_1, \end{aligned} \tag{a.8}$$

$$-m_2' = \{(1+\lambda)\, b_2(R_2)\, f_2(R_2 - y_1 + P_1)) - \lambda[1 - F_2(R_2 - y_1 + P_1))\} f_1. \tag{a.9}$$

Suppose the incentive constraint (a.1) is not binding in an interval (x, y). Then $T = 0$ in (x, y). By (a.6) and (a.7), $m_t = 0$ and $m_t' = 0$ in (x, y) for $t = 1, 2$. Now (a.8) and (a.9) become the same as (a.3) and (a.4) with q, r_1, and r_2 replaced by Q, R_1, and R_2. By using the same arguments as in the proof of Proposition 4, we can show that $P_1 - y_1 + R_2$ must be the constant k. But this implies $P_1' + R_2' = 1$ and contradicts (a.1). Q.E.D.

Proof of Proposition 6. The first order condition for D_1 is $D_1 = 0$ if and only if

[illegible] $\int^{r_2 - y_1 + p_1}$ [illegible]

$$\begin{aligned} &-\lambda \left\{ \int_0^{r_2 - (y_1 - p_1)} [y_2 + (y_1 - p_1) - b_2(y_2 + (y_1 - p_1))]\, dF_2 \right. \\ &\left. + r_2[1 - F_2(r_2 - (y_1 - p_1))] \right\} \\ &+ \lambda P_1 - \int_0^{R_2 - (y_1 - P_1)} b_2(y_2 + y_1 - P_1)\, dF_2 \end{aligned}$$

$$+\lambda\left\{\int_0^{R_2-(y_1-P_1)}[y_2+y_1-P_1-b_2(y_2+y_1-P_1)]\,dF_2\right.$$

$$\left.+R_2[1-F_2(R_2-(y_1-P_1))]\right\}\geqslant 0.$$

In deriving L_D from (a.2), we have left out wealth constraints (1) and the constraints involving the control variables u_1 and u_2 because their contribution to the Lagrangian L is always zero when they are multiplied by their multipliers and because they are irrelevant here.

$L_D(y_1)$ can be viewed as the net benefit of making y_1 a non-verified state. For the optimal contract, y_1 should be a non-verified state if and only if the benefit is non-negative. We want to see how the net benefit varies as y_1 increases. We have

$$\begin{aligned}dL_D/dy_1 &= (1+\lambda)b_1'+L_p p_1'+L_r r_2'-(L_p+\lambda)-L_P P_1'-L_R R_2'+(L_P+\lambda)\\ &= (1+\lambda)\,b_1'+L_P\\ &= (1+\lambda)\,b_1'\,b_1'+(1+\lambda)\,b_2(R_2)\,f_2(R_2-y_1+P_1))\\ &\quad -\lambda[1-F_2(R_2-y_1+P_1)],\end{aligned}$$

where L_p, L_r, L_P, and L_R are the partial derivatives of L with respect to p_1, r_2, P_1, and R_2, respectively, with f_1 omitted. Note that in L_D, y_1 always occurs with either p_1 or P_1 except in the terms $\lambda(p_1-b_1)$ and $\lambda(P_1-b_1)$. This is why we have $(L_p+\lambda)$ and $(L_P-\lambda)$. In simplifying the above expression, expression, I used the first order conditions $L_p=0$, $L_r=0$, and $P_1'+R_2'=0$, and the condition $L_P=L_R$ (with $Q(P_1-y_1)$ omitted).

Now suppose $D_1=0$ on $(v-e, v]$ and $D_1=1$ on $(v, v+e)$. By Propositions 2 and 4, we have $P_1(v)+R_2(v)-v>p_1(x)+r_2(x)-x=k$, where x is in $(v, v+e)$. That is, the firm prefers to be verified (the net liability $r_2-(y_1-p_1)$ is lower if verified). Since k is defined by $(1+\lambda)\,b_2(k)=\lambda[1-F_2(k)]/f_2(k)$ and since b_2 is non-decreasing and $[1-F_2]/f_2$ is decreasing, we have

$$\begin{aligned}&(1+\lambda)\,b_2(P_1(v)+R_2(v)-v)\\ &\qquad>\lambda[1-F_2(P_1(v)+R_2(v)-v)]/f_2(P_1(v)+R_2(v)-v).\end{aligned}$$

This implies that $(1+\lambda)\,b_2(R_2(v))\,f_2(P_1(v)+R_2(v)-v)>\lambda[1-F_2(P_1(v)+R_2(v)-v)]$ because $R_2(v)+P_1(v)-v\geqslant R_2(v)$, $f_2>0$, and b_2 is non-decreasing.

So we have proved $dL_D(v)/dy_1>0$. Since $L_D(v)\geqslant 0$ by assumption, we must have $L_D(y_1)>0$ for y_1 in $(v, v+e)$ for some $e>0$. This means that we should have $D_1=0$ in $(v, v+e)$. This is a contradiction. Q.E.D.

Proof of Proposition 7. Since all constraints are linear in the control variables u_t and the integrand of the objective function does not depend on the control variables, and since y_I is bounded in $[L_1, H_I]$, to apply Cesari's existence theorem [1], all we need is the boundedness of the state variables p_1, r_2, P_1, and R_2. They are not bounded below as stated in (1).

Consider a version of (5) by adding a lower bound B for P_1, R_2, p_1, and r_2. Now Cesari's theorem [1] implies that an optimal solution exists. Note that if B is low enough, nowhere in the proofs of the previous propositions will we be constrained by the fact that p_1 and P_1 have a lower bound. That is, as long as B is low enough, the optimal solution will not be bounded below and it will not be affected by the lower bound. When we take the limit as B approaches to $-\infty$, an optimal solution exists for each B and it does not depend on B. Therefore this optimal solution must be an optimal solution to (5). Q.E.D.

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