Consumption and Bubbles

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Some studies characterize bubbles as speculative phenomena in which investors pay more than the value of the asset's dividend stream in anticipation of receiving a profit by selling the asset later. We study the number of *consumption* opportunities as a necessary condition for the existence of bubbles. Our model permits continuous trading over a finite horizon, but it also assumes investors consume at discrete dates. Our main result is that when investors appropriately prefer more consumption to less, there are no bubbles on the prices of those assets that contribute to the aggregate financial wealth if the number of consumption opportunities is uniformly bounded across states of nature. This result clearly identifies market clearing as an additional restriction that helps to determine the martingale properties of equilibrium asset prices.

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1   Introduction

In neoclassical economics, an asset pricing bubble exists if the price of an asset exceeds the lowest cost of superreplicating its future dividends. Tirole (1982) shows a necessary condition for bubbles is that such investors collectively have an unbounded number of opportunities to trade. Some studies therefore characterize bubbles as speculative phenomena in which investors pay more than the value of the asset's dividend stream in anticipation of receiving a profit by selling the asset later. Models of bubbles typically use discrete-time and infinite-horizons, but some use the continuous-time and finite-horizons. Both types provide an unbounded number of trade opportunities.

We study the number of *consumption* opportunities as a necessary condition for the existence of bubbles. Our model permits continuous trading over a finite horizon, but it also assumes investors consume at discrete dates. This gives natural flexibility in choosing the number of consumption dates to be, for example, uniformly bounded across states, finite almost surely, or infinite. We also permit arbitrary feasible portfolios to serve as numeraires for constraints on negative wealth. This is the first study to examine asset pricing bubbles in which numbers of the trading and consumption dates may have di erent degrees of finiteness, and to examine the impact of numeraires for constraints on equilibrium bubbles.

Our main result is that when investors appropriately prefer more consumption to less, there are no bubbles on the prices of those assets that contribute to the aggregate financial wealth if the number of consumption opportunities is uniformly bounded across states of nature. There are also no bubbles on the numeraire portfolios. Our results clearly reflect market clearing as an equilibrium restriction that helps to determine the martingale properties not only for the assets in positive net supply but also for assets or portfolios used as numeraires. The result augments the usual partial equilibrium studies by helping to identify the extent to which limited arbitrages, equivalent local martingale measures, and other "local martingale properties" of asset prices play a role in equilibrium.

We use the idea that bubbles on the prices of the positive net supply assets must be accompanied by an appropriate accumulation of financial wealth, and this accumulation of financial wealth must be consistent with both market clearing and optimal portfolio choice when all investors prefer more consumption to less. In particular, there cannot be a bubble on the price of a positive net supply asset if there is *any* portfolio whose value both dominates every investor's financial wealth and represents the lowest cost of superreplicating its payouts. But with a uniformly bounded number of consumption dates, such a portfolio automatically exists if investors appropriately prefer more to less. Every investor's net consumption at each date is the maximum amount of consumption that can be gotten through trade, and this cannot exceed the cum-dividend price of the market portfolio and the amount of negative wealth permitted to all investors. As the sum of these present values is finite when summed over investors and a uniformly bounded number of consumption dates, there can be

no bubbles on the prices of assets in positive net supply.[1]

Our bound on aggregate financial wealth is automatic, and exists even if markets are incomplete and the aggregate endowment has infinite present value.[2] The studies of the equilibrium properties of bubbles by Santos and Woodford (1997) and Loewenstein and Willard (2000b) assume the aggregate endowment has finite present value. This assumption allows an arbitrary number of consumption dates, but it does require assuming something about a quantity endogenous to an equilibrium.

Section 5.3 presents an example of an equilibrium that not only highlights the role of the number of consumption dates but also has features that, to our knowledge, are the first of their kind.[3] In the example, the final date of consumption is known, but no investor is sure he whether he will consume at the final date or beforehand. This creates a model in which the number of consumption dates is almost surely finite but not uniformly bounded across paths. There is a bubble on the equilibrium price of the positive net supply asset. The bubble has a finite lifespan and is uniformly bounded; in fact, the asset's price itself is uniformly bounded. Both equilibrium consumption and consumption net of endowments are uniformly bounded.

We begin our analysis with a model having only one consumption date. Section 2 describes this model. Section 3 and Section 4 develop economic and mathematical results within the context of the one-period model. Through a transformation, these results extend to our general multiperiod consumption model. Our main results about the multiperiod model and our example are presented Section 5. Section 6 presents direct applications of our main results to issues arising in the literature of pricing by equivalent martingale measures.

2   Model with One Consumption Date

We begin our study of equilibrium asset pricing bubbles by describing the continuous-time model we use. While our study of consumption and pricing bubbles will ultimately allow consumption to take place at a potentially unbounded and random number of dates, we introduce the economics of our main results using the special case in which consumption takes place only one deterministic date. The multiperiod consumption model appears in Section 5, and our results there will rely on our one-period analysis through a transformation.

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[1]Our analysis permits incomplete markets, which complicates the definition of present value because there can be infinitely many stochastic discount factors. As we prove, an interpretation of finite present value is that one can construct a portfolio that both requires finite initial investment and has payouts that superreplicates the quantity of interest.

[2]An interpretation of infinite present value for the aggregate endowment is that one cannot construct a portfolio that both requires finite initial investment and has payouts that superreplicate the aggregate endowment.

[3]Our example modifies an unpublished example we distributed in papers with various titles. We also use some of the mathematical structure developed for a partial equilibrium example by Delbaen and Schachermayer (1998).

The important features of the model are asset prices, investors' consumption-investment choice problems (including preferences and wealth constraints), and financial market equilibrium. We now describe these features.

2.1 Assets

Trade takes place on a time interval $[0, T]$, where T is finite and deterministic. Uncertainty is represented by an underlying complete probability space with a probability measure P and a standard d-dimensional Brownian motion Z, and information arrival is described by the completed filtration $\{\mathcal{F}_t : t \in [0, T]\}$ generated by Z.

There are $K + 1$ long-lived financial assets. Dividend payments occur at only date T. The first asset is a locally riskless bond having a price process B that is strictly positive, predictable, and finite. We assume B has finite variation and satisfies

$$B(t) = 1 + \int_0^t r(s)B(s)ds$$

for a predictable locally riskless rate r. The remaining K assets are locally risky and have nonnegative prices described by

$$S_k(t) = S_k(0) + \int_0^t \mu_k(s)S_k(s)ds + \sum_{j=1}^d \int_0^t \sigma_{kj}(s)S_k(s)dZ(s)$$

for $k = 1, \ldots, K$.[4] The processes μ_k and σ_{kj} are finite, progressively measurable, and satisfy the usual integrability conditions making the stochastic integrals well-defined (see Karatzas and Shreve (1988, Chapter 3.2)). Our analysis permits incomplete markets and locally redundant asset returns. We interpret $S^k(T)$ as the liquidating dividend of asset k (i.e., $S^k(T)$ is its cum-dividend price), which will be appropriate given our later assumptions about investor preferences and market clearing.

2.2 Investors

There is a finite number I of investors indexed by i. Each investor i receives an initial endowment of $\pi_B^i(0)$ shares of the bond and $\pi_S^i(0)$ shares of the risky assets, giving initial financial wealth $w^i = \pi_B^i(0)B(0) + \pi_S^i(0)S(0)$. Each investor i receives a nonnegative terminal private endowment, denoted $e^i(T)$, and consumes an amount $c^i(T)$ at date T equal to this endowment plus the terminal financial wealth gotten from trading.

Each investor may trade continuously on $[0, T]$. Denote a trading strategy by $\pi^i = (\pi_B^i, \pi_S^i)$, where $\pi_B^i(t)$ represents shares of the bond and the K-dimensional row vector $\pi_S^i(t) = [\pi_1^i, \ldots, \pi_k^i, \ldots, \pi_K^i]$ represents shares of the risky assets, both at time t.

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[4]This specification implies asset prices have almost surely continuous paths, which reduces notation. Our main results would be unchanged for more general specifications given our assumed information structure.

Given initial financial wealth w^i, every investor i's financial wealth process W^i must satisfy the budget equation

$$W^i(t) \equiv \pi_B^i(t)B(t) + \pi_S^i(t)S(t) = w^i + \int_0^t \pi_B^i(s)dB(s) + \int_0^t \pi_S^i(s)dS(s)$$

$$\text{and} \quad W^i(T) \geq -e^i(T). \tag{2.1}$$

The budget equation makes financial wealth "self-financing" and enforces nonnegative consumption $c^i(T) \equiv W^i(T) + e^i(T)$. We assume π^i is progressively measurable, is locally bounded, and makes the stochastic integrals describing portfolio gains in (2.1) well-defined (see Karatzas and Shreve (1988, Chapter 3.2)).

Each investor i has utility $U^i(W^i(T))$ for terminal financial wealth $W^i(T)$. We assume U^i is a real-valued functional defined over all $\mathcal{F}_T$-measurable real-valued random variables W satisfying $W \geq -e^i(T)$. Our results sometimes assume investors prefer more terminal wealth to less, as we now define.

Definition 2.1. *Investor i **prefers more terminal wealth to less** if $U^i(W + Y) > U^i(W)$ for any $\mathcal{F}_T$-measurable real-valued random variables W and Y satisfying $P(W \geq -e^i(T)) = 1$, $P(Y \geq 0) = 1$, and $P(Y > 0) > 0$.*

Our results later assume at least one investor has "regular" preferences so that a particular type of approximate arbitrage will be inconsistent with optimal choice, as we will describe in Section 3. Here is our definition of "regular."

Definition 2.2. *Investor i's preferences U^i are **regular** if (i) the investor prefers more terminal wealth to less, and (ii) for any $\mathcal{F}_T$-measurable finite-valued random variable W satisfying $P(W \geq -e^i(T)) = 1$ there exist real-valued sequences $\epsilon_n \downarrow 0$ and $\delta_n \uparrow \infty$ such that*

$$(\exists n) \quad U^i\big((1 - \epsilon_{n^*})W + \delta_{n^*}Y_{n^*}\big) > U^i(W)$$

where $\{Y_n\}$ is a sequence of nonnegative $\mathcal{F}_T$-measurable random variables that almost surely converges to Y, where $P(Y \geq 0) = 1$ and $P(Y > 0) > 0$.

Linear terminal utility, unbounded expected utility preferences, and expected utility functions continuous in terminal wealth are examples of regular preferences.[5]

2.3   Wealth Constraints and Portfolio Choice

We now present the investors' portfolio choice problems. An important feature of choice in continuous-time models is a constraint on negative wealth that serves to make "doubling strategies" infeasible at some scale (Harrison and Pliska, 1981; Dybvig and Huang, 1988). Such constraints are typically either "endogenous" or "exogenous,"

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[5]Loewenstein and Willard (2000a), who use a similar definition for the same purpose we do, present an example of preferences that would prefer more terminal wealth to less but are not regular according to our definition.

the meanings of which are described by the following choice problems. Our framework allows both types of constraints, and allows us to compare the potentially different asset pricing implications of each type.

We use the notation a^i for a process that describes lower bound on the wealth process of investor i. Here is our main assumption about each a^i.

Assumption 2.1. *A bound on negative wealth a^i is pathwise nonpositive $(P(\forall t \in [0, T] \ a^i(t) \leq 0) = 1)$ and is the value of a self-financing portfolio; that is,*

$$a^i(t) = \alpha_B(t)B(t) + \alpha_S(t)S(t) = a^i(0) + \int_0^t \alpha_B(s)dB(s) + \int_0^t \alpha_S(s)dS(s) \quad (2.2)$$

for an adapted portfolio $\alpha = (\alpha_B, \alpha_S)$. We assume α is locally bounded and satisfies conditions ensuring the stochastic integrals in (2.2) are well-defined.

One traditional type of wealth constraint is the *exogenous* constraint, described by the following consumption-investment choice problem.

Choice Problem 2.1 (Exogenous Wealth Constraint). *Given initial wealth w^i and a fixed constraint on negative wealth a^i satisfying Assumption 2.1, choose a portfolio π^i to maximize the utility $U^i(W^i(T))$ subject to the budget constraint (2.1) and*

$$P\big((\forall t \in [0, T]) \ W^i(t) \geq a^i(t)\big) = 1. \quad (2.3)$$

The constraint (2.3) on negative wealth applies to every portfolio that investor i might choose. The interpretation is a monitoring agency or trading partners determine an investor's creditworthiness and limit negative wealth by monitoring wealth as it evolves (Dybvig and Huang, 1988; Magill and Quinzii, 1994; Loewenstein and Willard, 2000a). Dybvig and Huang (1988) and Loewenstein and Willard (2000a) consider the special cases of nonnegative wealth ($a \equiv 0$) and negative wealth bounded by units of the bond ($W(t) \geq a(t) \equiv -\gamma B(t)$ for fixed $\gamma > 0$). Other special cases include $a(t) \equiv -\gamma S_k(t)$ and $a(t) \equiv -\gamma \bar{\pi}_S S(t)$ for fixed $\gamma > 0$, which would limit negative wealth using a particular asset or the market portfolio as a numeraire.

Delbaen and Schachermayer (1994, 1995, 1997a,b) use a different type of constraint that allows investor i to choose a lower bound for negative wealth simultaneously with a portfolio. Delbaen and Schachermayer's results are also connected to bubbles, so we include their *endogenous* constraints in our study. These constraints appear in the following choice problem.

Choice Problem 2.2 (Endogenous Wealth Constraints). *Given initial wealth w^i and a collection $\mathcal{A}^i$ of constraints on negative wealth, choose a portfolio π^i and an $a^i \in \mathcal{A}^i$ to maximize the utility $U^i(W^i(T))$ subject to the budget constraint (2.1) and*

$$P\big((\forall t \in [0, T]) \ W^i(t) \geq a^i(t)\big) = 1. \quad (2.4)$$

We make the following assumption to ensure endogenous wealth constraints can be scaled arbitrarily which distinguishes them economically from exogenous constraints.

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Assumption 2.2. $\mathcal{A}^i \neq \{0\}$. *Each $a \in \mathcal{A}^i$ satisfies Assumption 2.1 and*

$$a_1, a_2 \in \mathcal{A}^i \implies a_1 + a_2 \in \mathcal{A}^i. \tag{2.5}$$

The enforcement of endogenous constraints relies on an investor's perception of a limit on negative wealth (Magill and Quinzii, 1994), but does not fix in advance a specific lower bound. Every feasible strategy can be scaled by integer amounts in Problem 2.2 since $na \in \mathcal{A}^i$ if $a \in \mathcal{A}^i$. Delbaen and Schachermayer (1994, 1995, 1997a,b) study the special case $\mathcal{A}^i = \{a(t) = -\gamma B(t) : \gamma \in \Re_+\}$, which uses the locally riskless bond as the numeraire. It is important to note that a solution to Problem 2.2 includes both a specific trading strategy and a specific lower bound on wealth.

2.4 Equilibrium

We ultimately identify some necessary properties of equilibrium prices given exogenous and endogenous bounds on negative wealth. Here we define equilibrium. Denote the net supplies of assets by $\bar{\pi}_B = \sum_{i=1}^I \pi_B^i(0)$ and $\bar{\pi}_S = \sum_{i=1}^I \pi_S^i(0)$, and assume these quantities are nonnegative and constant. If $\bar{\pi}_k > 0$, then we say asset k is in positive net supply. We assume that locally riskless borrowing and lending o set each other so that the bond is in zero net supply ($\bar{\pi}_B = 0$).

Definition 2.3. *An **equilibrium** consists of asset prices (B, S) satisfying the assumptions of Section 2 and self-financing portfolios $\{\pi^i : i = 1, \ldots I\}$ such that:*

1. *Given the asset prices, every investor i's portfolio solves Problem 2.1 or Problem 2.2 given Assumptions 2.1 and 2.2 and initial wealth $w^i = \pi_S^i(0)S(0) + \pi_B^i(0)B(0)$,*

2. *The asset markets clear:*

$$\bar{\pi}_B = \sum_{i=1}^I \pi_B^i(t) = 0 \quad and \quad \bar{\pi}_S = \sum_{i=1}^I \pi_S^i(t) \quad at\ all\ times\ t \in [0, T], \tag{2.6}$$

   and

3. *The consumption market clears:*

$$\sum_{i=1}^I W^i(T) = \bar{\pi}_S S(T).$$

Given market clearing for the consumption good, our assumption that every investor prefers more to less implies the aggregate terminal value of the risky assets in positive net supply is equal to the aggregate liquidating dividend of the assets.

Remark 2.1. *Within a given equilibrium, each investor i's constraint on negative wealth a^i can be regarded as fixed. This is automatic for exogenous constraints. Even for endogenous constraints, the existence of an equilibrium would require every investor to have an optimal portfolio and a corresponding $a^i \in \mathcal{A}^i$*

Proof. See Appendix A.2. □

Second we identify a class of processes to describe a notion of "state prices" for incomplete markets. Given a $\tilde{\theta}$ of Proposition 3.1, let $\theta(t)$ be its orthogonal projection onto the range space of $\sigma(t)$ for all $t \in [0, T]$, where prime denotes transpose. Karatzas and Shreve (1998, Lemma 1.4.4) show θ is progressively measurable. Let τ be the stopping time defined by

$$\tau = \inf \left\{ t \in [0, T] \,|\, \int_0^t ||\theta(s)||^2 ds = \infty \right\}$$

where $\tau = \infty$ if such a t does not exist. We assume $P(\tau = 0) = 0$. Let $\mathcal{V}$ denote the set of progressively measurable d-dimensional processes ν with $\sigma(t)\nu(t) = 0$, Lebesgue $\otimes P$ a.s., and $\int_0^T ||\nu(s)||^2 ds < \infty$, P-a.s. For a given $\nu \in \mathcal{V}$, define

$$\rho(t) = \frac{\exp\left(-\int_0^t \big(\theta(s) + \nu(s)\big) dZ(s) - \int_0^t ||\theta(s) + \nu(s)||^2 ds\right)}{B(t)} \tag{3.8}$$

on $\{t < \tau\}$ and $\rho(t) = 0$ on $\{t \geq \tau\}$. Every ρ is nonnegative and continuous. In this context, financial markets are complete if $\mathcal{V} = \{0\}$.

The processes ρ, $\nu \in \mathcal{V}$, might be called "state price densities." However, unlike for finite-dimensional models, the existence of an optimum for an investor who prefers more to less does not guarantee any particular ρ, especially ρ^0, is strictly positive. In the present model, ρ^0's hitting zero with positive probability can be thought of as having an investment in the optimal growth portfolio converge to infinity on the set $\{\rho^0(T) = 0\}$. This would introduce an approximate arbitrage (which Loewenstein and Willard (2000a) call a "cheap thrill") in which no feasible strategy is guaranteed to improve utility but the limit of a sequence of strategies does for an investor with regular preferences (Definition 2.2).[6] We now state the formal result.

Proposition 3.2. *Suppose an optimal portfolio exists for Problem 2.1 or Problem 2.2 for some investor who has regular preferences and positive initial wealth $w^i > 0$. Then $P\big(\rho^0(T) > 0\big) = 1$ and, consequently, $P\big(\rho(T) > 0\big) = 1$ for all $\nu \in \mathcal{V}$.*

Proof. See Appendix A.2. □

Given our prior assumptions, state prices in any equilibrium of our model cannot hit zero with positive probability in an equilibrium provided some investor has regular preferences. To avoid having to deal with the possibility of zero state prices in our study of bubbles, we make the following standing assumption.

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[6]In discrete-state models, zero state prices imply the existence of a zero-cost arbitrage inconsistent with optimal choice (Dybvig and Ross, 1987). But Loewenstein and Willard (2000a) show in continuous-time models that zero state prices might imply only approximate arbitrages, namely, strategies that require vanishingly positive initial investment, maintain nonnegative wealth, and generate large payoﬀs on $\{\rho^0(T) = 0\}$. We relax Loewenstein and Willard's assumption that σ is invertible by using a diﬀerent set of strategies to construct a cheap thrill, namely stopped investments in the optimal growth portfolio.

Assumption 3.1. *An optimum exists for Problem 2.1 or Problem 2.2 for some investor who has regular preferences and is endowed with positive initial wealth.*

Given Assumption 3.1, we can link the ρ 's to the replicating costs of streams of payouts given bounds on negative wealth. The next result extends Loewenstein and Willard (2000a, Proposition 3.1) to include incomplete markets and more general types of bounds on negative wealth.

Proposition 3.3. *Given Assumption 3.1 and a nonpositive process a describing a bound on negative wealth that satisfies Assumption 2.1,*

1. *If a self-financing trading strategy generates a wealth process that satisfies $W(t) \geq a(t)$ pathwise on $[0, T]$, then*

$$(\forall \nu \in \mathcal{V}) \qquad E\big[\rho\ (T)W(T)\big] \leq w - \Big(a(0) - E\big[\rho\ (T)a(T)\big]\Big). \qquad (3.9)$$

2. *If a $\mathcal{F}_T$-measurable random variable X satisfies $X \geq a(T)$ and*

$$\sup_{\mathcal{V}} E\big[\rho\ (T)\big(X - a(T)\big)\big] = w - a(0), \qquad (3.10)$$

then there is a self-financing trading strategy with wealth W that satisfies $W(0) = w$, $W(T) \geq X$, P-almost surely, and $W(t) \geq a(t)$ pathwise on $[0, T]$.

Proof. See Appendix A.2. □

We say the portfolio in Proposition 3.3 (2) "superreplicates" the payout X given the constraint a on negative wealth. We now use Proposition 3.3 to study asset pricing bubbles and limited arbitrage.

3.2 Bubbles and Limited Arbitrage

The Law of One Price says two portfolios having the same payouts have the same price. A violation of the Law of One Price is often associated with an asset pricing bubbles, which we now define.

Definition 3.1 (Asset Pricing Bubble). *An asset's price has a **bubble** if it exceeds the lowest cost of superreplicating the asset's future dividends with a portfolio that maintains nonnegative wealth.*

A bubble might seem to be inconsistent with optimal choice since it creates an arbitrage opportunity. Arbitraging a bubble involves short selling the higher-cost asset and buying the lower-cost superreplicating portfolio. However, the feasibility of these arbitrages depends on the nature of the bubble and on the investors' bounds on negative wealth, as we describe in this section.

The potential for bubbles in the partial equilibrium setting arises from the fact that optimal choice generally requires only that $\rho\, B$, $\rho\, S$, and $-\rho\, a^i$ be nonnegative local martingales (supermartingales) for each $\nu \in \mathcal{V}$.[7] This implies that $\forall \nu \in \mathcal{V}$,

$$B(0) \geq E\left[\rho\,(T)B(T)\right], \quad S(0) \geq E\left[\rho\,(T)S(T)\right],$$
$$\text{and} \quad a^i(0) \leq E\left[\rho\,(T)a^i(T)\right]. \quad (3.11)$$

An inequality in (3.11) in strict if and only if the corresponding process is not a martingale. Given a strict inequality, a bubble generally arises because the final payout can be superreplicated at a cost lower than the initial value, as we show.

Proposition 3.4. *Given Assumption 3.1, the inequality*

$$S^k(0) > \sup_V E\left[\rho\,(T)S^k(T)\right] \quad (3.12)$$

implies there is a self-financing trading strategy that requires initial wealth $w < S^k(0)$, generates a payoff $W(T) \geq S^k(T)$, and maintains pathwise nonnegative wealth. The righthand side of inequality (3.12) is the lowest cost of superreplicating asset k's dividend $S^k(T)$ given pathwise nonnegative wealth.

Proof. The proof follows directly from Proposition 3.3: take $a \equiv 0$, $X \equiv S^k(T)$, and $w = \sup_V E\left[\rho\,(T)S^k(T)\right]$. $\qquad \square$

An analogous result holds for the locally riskless bond price B when

$$B(0) > \sup_V E\left[\rho\,(T)B(T)\right]. \quad (3.13)$$

The strict inequality in (3.12) implies the asset's price has a bubble; strict inequality in (3.13) implies the bond's price has a bubble. Loewenstein and Willard (2000a,b) and Heston, Loewenstein, and Willard (2007) give several explicit closed-form examples of both types of bubbles that are consistent with optimal portfolio choice and strictly monotone preferences for finite-horizon continuous-time finite-horizon models.

When the strict inequality

$$a^i(0) < E\left[\rho\,(T)a^i(T)\right]$$

holds, then $\rho\, a^i$ is not a martingale. This strict inequality is related to "limited arbitrage" and bubbles in the portfolios that serve as numeraires for the constraints on negative wealth, as we now show.

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[7]A process X is a local martingale if there exists an increasing sequence of stopping times $\{_n\}$ such that $\lim_{n\to\infty}{}_n = T$ almost surely and each stopped process $X(t \wedge {}_n)$ is a martingale. A nonnegative local martingale is a supermartingale (Karatzas and Shreve, 1988, Exercise 1.5.19). Strict local martingales are "explosive" on small probability sets in that they satisfy both $E[\max_{t\in[0,T]} {}^\nu(t)S^k(t)] = \infty$ and $P(\max_{t\in[0,T]} {}^\nu(t)S^k(t) \geq\ ) \leq S^k(0)/$ (Protter (1992, Theorem I.47) and Revuz and Yor (1994, Theorem II.1.7)).

Proposition 3.5. *Suppose Assumption 3.1 holds, and let a be a given constraint on negative wealth satisfying Assumption 2.1. Then*

$$(\forall \nu \in \mathcal{V}) \quad a(0) \le E\big[\rho\ (T)a(T)\big] \le 0. \tag{3.14}$$

Moreover, the inequality

$$a(0) < \inf_{V} E\big[\rho\ (T)a(T)\big]. \tag{3.15}$$

holds if and only if there is a self-financing portfolio with that requires no initial wealth ($w = 0$), provides an arbitrage profit ($P(W(T) > 0) > 0$ and ($P(W(T) \ge 0) = 1$), honors the constraint on negative wealth ($P((\forall t \in [0,T])\ W(t) \ge a(t)) = 1$), but risks temporary negative wealth ($P((\exists t \in [0,T))\ W(t) < 0) > 0$).

Proof. See Appendix A.2. □

If inequality (3.15) holds, then the payout $-a(T)$ can be superreplicated at a cost lower than $-a(0)$, the initial wealth required by the portfolio that defines the negative wealth constraint. Thus the portfolio involves bubbles. This might be because one or more of the assets in the portfolio has a bubble, or because the portfolio itself throws away wealth by following a suicide strategy (see, e.g., Harrison and Pliska (1981)). An arbitrage that buys the superreplicating strategy for $-a(T)$ and shorts the portfolio describing the bound on wealth is automatically feasible if a is exogenous and at all scales if a is endogenous. Loewenstein and Willard (2000a) present examples of limited arbitrages and optimal solutions given an exogenous constraint (Problem 2.1), even for an investor who has regular preferences.

A solution for Problem 2.2 does not allow strategies like those in Proposition 3.5 the endogenous wealth constraints can be arbitrarily scaled (Assumption 2.2). This a solution for Problem 2.2 requires the equality

$$(\forall a \in \mathcal{A}^i) \quad a(0) = \inf_{V} E\big[\rho\ (T)a(T)\big]. \tag{3.16}$$

Equality (3.16) is not necessary for a solution to Problem 2.1 because the fixed bounds on negative wealth limit the scale of the arbitrage. The di erent implications of these assumptions about the nature of lower bounds on wealth explain the di erent conclusions about equivalent martingale measures by Delbaen and Schachermayer (1994) and Loewenstein and Willard (2000a), as Section 6 explains.

We now turn to restrictions implied by the existence of an equilibrium.

4   Equilibrium with One Consumption Date

Above we demonstrate that bubbles and limited arbitrage are potential "partial equilibrium" properties consistent with optimal portfolio choice and constraints on negative wealth. We now show, however, these properties are inconsistent with an equilibrium – most particularly, market clearing – for models with one consumption date.

We remind the reader of our main assumptions, which we assume throughout Section 4.

Assumption 4.1. *We assume the constraints on negative wealth are described by the nonpositive values of self-financing portfolios as in Assumption 2.1. For endogenous constraints, we additionally assume each set $\mathcal{A}^i$ has the scaling property in Assumption 2.2. We assume every investor chooses an optimal portfolio for Problem 2.1 or Problem 2.2.*

We will also assume some investor has regular preferences and positive initial wealth to rule out zero state prices (Assumption 3.1), but we will include this assumption directly in the statements of our following results.

Here is our main equilibrium result.

Theorem 4.1. *Assume an equilibrium exists when every investor prefers more terminal wealth to less and some investor has regular preferences and positive initial wealth. Then there is a $\nu^* \in \mathcal{V}$ such that:*

1. If $\bar{\pi}_k > 0$, then

$$S^k(0) = \sup_V E\left[\rho^*(T)S^k(T)\right] = E\left[\rho^*(T)S^k(T)\right],$$

and $S^k(0)$ is the lowest cost of replicating the dividend $S^k(T)$ given pathwise nonnegative wealth. That is, the equilibrium prices of the positive net supply assets do not have bubbles.

2. For every investor i, the equality

$$a^i(0) = \sup_V E\left[\rho^*(T)a^i(T)\right] = E\left[\rho^*(T)a^i(T)\right] \tag{4.17}$$

holds for the exogenous constraint on negative wealth a^i in Problem 2.1. For endogenous constraints, equality (4.17) holds for every $a^i \in \mathcal{A}^i$ in Problem 2.2. Thus there are no limited arbitrage opportunities for both exogenous and endogenous constraints on negative wealth.

Theorem 4.1 rules out bubbles that would affect equilibrium aggregate financial wealth (the value of the positive net supply assets), as well as bubbles that would affect any investor's constraint on negative wealth. We explain below that such bubbles are impossible in our model because every investor's equilibrium consumption net of private endowments is limited by the aggregate terminal dividend plus the absolute value of the aggregate allowable negative terminal financial wealth of all investors.[8]

We now describe the economic steps of our proof (the more mathematical details appear in the Appendix). Our assumption that all investors prefer more terminal wealth to less implies they invest no more than necessary to finance consumption, as we now show.

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[8]In contrast to previous studies, our Theorem 4.1 uses neither explicit nor implicit assumptions about the value of the aggregate endowment, as we explain in more detail in Section 5.2.

Proposition 4.1. *Assume an equilibrium exists when every investor prefers more terminal wealth to less and some investor has regular preferences and positive initial wealth. Given either an exogenous constraint a^i on negative wealth in Problem 2.1 or the equilibrium endogenous constraint a^i in Problem 2.2, each investor i's wealth satisfies*

$$W^i(t) - a^i(t) = \operatorname{essup} \ _V E_t\left[\frac{\rho(T)}{\rho(t)}\left(W^i(T) - a^i(T)\right)\right], \qquad (4.18)$$

where essup *denotes essential supremum.*[9]

Proof. See Appendix A.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The presence of a^i in (4.18) might seem unusual, as the more typical versions of static budget equations would have a^i be identically zero in (4.18). Recall, however, we have not yet ruled out the possibility a limited arbitrage opportunity exists (see Section 3.2). Undertaking a limited arbitrage opportunity would reduce the cost of a given consumption plan compared to, say, what it would cost by maintaining nonnegative wealth. An investor who prefers more to less would exploit a limited arbitrage opportunity allowed by an exogenous lower bound on wealth, and the presence of a^i in (4.18) reflects this.[10]

Given that monotone investors finance their consumption at the lowest cost, we now use market clearing and the constraints on negative wealth to establish an *upper* bound on each investor's wealth. These upper bounds will show that aggregate financial wealth cannot grow large enough to support bubbles on assets that contribute to it (those assets in positive net supply). Other studies, notably Santos and Woodford (1997) and Loewenstein and Willard (2000b), make assumptions that give such bounds; however, our bound arises endogenously given one consumption date.

Because every investor's equilibrium financial wealth satisfies (4.18), clearing the consumption market bounds every investor's wealth by the largest possible value of the aggregate terminal dividends $\bar{\pi}_S S(T)$

inequalities:

$$W^i(t) \leq W^i(t) - a^i(t) = \operatorname*{essup}_{\nu} E_t\left[\frac{\rho_\nu(T)}{\rho_\nu(t)}\big(W^i(T) - a^i(T)\big)\right]$$

$$\leq \operatorname*{essup}_{\nu} E_t\left[\frac{\rho_\nu(T)}{\rho_\nu(t)}\sum_{i=1}^{I}\big(W^i(T) - a^i(T)\big)\right]$$

$$= \operatorname*{essup}_{\nu} E_t\left[\frac{\rho_\nu(T)}{\rho_\nu(t)}\Big(\bar{\pi}_S S(T) - \sum_{i=1}^{I} a^i(T)\Big)\right]. \quad (4.19)$$

Now we show that clearing the asset market bounds aggregate financial wealth in a manner that rules out bubbles on positive net supply assets. This bound relies on our assumption that the number of investors, I, is finite. Specifically, we have

$$0 \leq \sum_{i=1}^{I}\big(W^i(t) - a^i(t)\big) = \bar{\pi}_S S(t) - \sum_{i=1}^{I} a^i(t)$$

$$\leq I \times \operatorname*{essup}_{\nu} E_t\left[\frac{\rho_\nu(T)}{\rho_\nu(t)}\Big(\bar{\pi}_S S(T) - \sum_{i=1}^{I} a^i(T)\Big)\right]. \quad (4.20)$$

We state a useful mathematical result.

Lemma 4.1. *Let X be a nonnegative process such that $\rho_\nu X$ is a local martingale for all $\nu \in \mathcal{V}$. Suppose $X(t) \leq \operatorname*{essup}_{\nu} E_t\left[\frac{\nu(T)}{\nu(t)}\right]$ for some nonnegative $\mathcal{F}_T$-measurable random variable with $\sup_{\nu} E[\rho_\nu(T)] < \infty$. Then*

$$X(t) = \operatorname*{essup}_{\nu} E_t\left[\frac{\rho_\nu(T)}{\rho_\nu(t)} X(T)\right],$$

and there exists $\nu^* \in \mathcal{V}$ such that $\rho_{\nu^*} X$ is a martingale.

Proof. See Appendix A. □

Taking $= I \times (\bar{\pi}_S S(T) - \sum_{i=1}^{I} a^i(T))$ in Lemma 4.1, the present value of is finite since, by Proposition 4.1 and the definition of supremum, we have

$$\sup_{\nu} E[\rho_\nu(T)] = I \times \sup_{\nu} E\left[\rho_\nu(T)\Big(\bar{\pi}_S S(T) - \sum_{i=1}^{I} a^i(T)\Big)\right]$$

$$\leq I \times \sum_{i=1}^{I} \sup_{\nu} E\left[\rho_\nu(T)\big(W^i(T) - a^i(T)\big)\right] = I \times \sum_{i=1}^{I}(w^i - a^i(0)) < \infty.$$

Thus Lemma 4.1 gives a $\nu^* \in \mathcal{V}$ such that the process

$$\rho_{\nu^*}(t)\Big(\bar{\pi}_S S(t) - \sum_{i=1}^{I} a^i(t)\Big) \quad (4.21)$$

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is a martingale. As each element in this sum is a nonnegative supermartingale, the martingale property of (4.21) holds only if $\rho^* S^k$ is a martingale whenever $\bar{\pi}_k > 0$ and if $\rho^* a^i$ is a martingale for every equilibrium a^i. That (4.17) holds for all $a^i \in \mathcal{A}^i$ for endogenous constraints follows from inequality (3.16).

Theorem 4.1 immediately follows. Although bubbles and limited arbitrages are consistent with partial equilibrium in the model with one consumption date, in a general equilibrium they cannot appear on the prices of assets that affect aggregate financial wealth because they would be incompatible with market clearing. We now show that this extends to the multiperiod consumption model if the number of consumption dates is uniformly bounded.

5 Equilibrium with Multiple Consumption Dates

We now extend our results to the continuous trade model that allows the possibility of consumption at multiple discrete and random dates. Our objective is to show that, in this model, bubbles on the prices of the positive net supply assets requires a non-uniformly bounded number of consumption dates. Our assumptions about the multiperiod model appear in Section 5.1. In Section 5.2 we show the ideas from the model with just one consumption date remain valid when the number of dates is bounded by a constant and investors appropriately prefer more to less. Our proofs use a tranformation to apply the results of one-period model above. Section 5.3 presents an example of an equilibrium in which the prices of the positive net supply assets have bubbles – in this example, the number of consumption dates is almost surely finite, but is not uniformly bounded.

5.1 Multiperiod Consumption Model

Here are our assumptions about the multiperiod consumption model. The economy ends at the deterministic date $T < \infty$.[11] All uncertainty and information arrival are represented by the completed filtration of a standard Brownian motion. This assumption applies to both asset prices and to the potential uncertainty about the timing of consumption dates.

Consumption prior to date T occurs at dates that are potentially random. To describe these dates, we first let $\{t_n : n = 1, \ldots, \infty\}$ denote a strictly increasing sequence of stopping times, and define the random number N to be $\sup\{n : t_n < T\}$ if such an n exists and 0 if not. For now, we assume $N < \infty$ almost surely, and set $N + 1$ to be the number of consumption dates. We assume the last consumption date is time T, and accordingly redefine $t_{N+1} \equiv T$. The consumption and dividend dates along a given path ω are described by the finite sequence $\{t_1(\omega), \ldots, t_{N(\ )}(\omega), t_{N(\ )+1}(\omega) = T\}$.

---

[11] Infinite horizon models also allow bubbles that "pop" at $T = \infty$, but such models inherently involve infinitely many consumption dates. As we focus on a bounded number of dates, developing the added notation and assumptions that would be needed to handle infinite-horizon models would not be useful for our study.

This allows consumption at random dates and the possibility that the total number of such dates is unknown until time T. Section 4 addresses the special case of one consumption date, in which $t_1 \equiv T$ and $N \equiv 0$.

Asset prices have properties analogous to those in Section 4. The locally riskless bond is in zero net supply ($\bar{\pi}_B = 0$), pays no dividends, and has the price B satisfying

$$B(t) = 1 + \int_0^t r(s)B(s)ds$$

on $[0, T]$ for a predictable locally riskless rate r. The net supplies of the risky assets, denoted $\bar{\pi}_S$, are constant and nonnegative. The risky assets' dividends, described by a vector process D, are nonnegative. Let S^{ex} denote the vector process of the ex-dividend asset prices. The risky assets pay dividends only on consumption dates. The ex-dividend price of each risky asset k satisfies

$$S_k^{\text{ex}}(t) = S_k^{\text{ex}}(t_n) + \int_{t_n}^t \mu_k(s)S_k^{\text{ex}}(s)ds + \sum_{j=1}^d \int_0^t \sigma_{kj}(s)S_k^{\text{ex}}(s)dZ(s) - D_k(t_{n+1})1_{\{t=t_{n+1}\}}$$

on any random interval $(t_n, t_{n+1}]$ with $1 \leq n \leq N$. Defining $\mathcal{V}$ the same as in Section 3, it follows from a straightforward extension that

$$\rho(t)S_k^{\text{ex}}(t) + \sum_{n=1}^{N+1} \rho(t_n)D_k(t_n)1_{\{t_n \ t\}}$$

is a nonnegative local martingale, and therefore a supermartingale, for each $\nu \in \mathcal{V}$.

We assume at most a finite number I of investors actively participate in the market at any given date $t \in [0, T]$. These investors need not be the same across all consumption dates, which permits many overlapping generations models. Each investor i solves the following portfolio choice problem.

Choice Problem 5.1. *Given initial wealth w^i and a $\mathcal{F}_t$-adapted constraint on negative wealth a^i, choose a portfolio π^i to maximize the utility of consumption $U^i((c_{t_n}^i : n = 1, \ldots, N)$ subject to the budget equation that requires financial wealth W^i to satisfy $W^i(0) = w^i$ and*

$$W^i(t) = \pi_S^i(t)S^{ex}(t) + \pi_B^i(t)B(t) = W^i(0) + \int_0^t \pi_S^i(s)dS^{ex}(s)$$

$$+ \sum_{t_n \ t} \pi_S^i(t_n)D(t_n) + \int_0^t \pi_B^i(s)dB(s) + \sum_{t_n \ t}\left(e^i(t_n) - c^i(t_n)\right)$$

and

$$P\big((\forall t \in [0, T]) \ W^i(t) \geq a^i(t)\big) = 1. \tag{5.22}$$

The choice problem specifies an exogenous bound on negative wealth. We do not formally consider endogenous lower bounds in our multiple consumption date analysis,

but the extension of our main results in this section ought to be straightforward given our work in Section 4.

Equilibrium in any multiperiod model requires consistency between the investors' preferences and the timing of consumption and endowments. Specific models would include assumptions about the investors' preferences for consumption at particular dates and when investors participate in the markets. (Our example in Section 5.3 will be one such specific model.) To avoid limiting our analysis by making highly specific assumptions, we just assume the following general properties for the multiperiod model. Each property could be derived from more primitive assumptions about the economy (as we do for the one-period model and our example).

Assumption 5.1. *Here are our assumptions about the multiperiod consumption-investment choice problem.*

- *If an investor optimally consumes a nonzero amount at some date, that investor's utility is strictly increasing for consumption at that date. No investor receives endowments after the last date he consumes.*

- *The process ρ^0 satisfies $P\{\rho^0(T) = 0\} = 0$. As described in Section 3, this is necessary for the absence of approximate arbitrages constructed from investments in the optimal growth portfolio having payoffs that converge to infinity on $\{\rho^0(T) = 0\}$.[12]*

- *The bound on negative wealth described by a^i is such that the process $-\rho\, a^i$ is a supermartingale over the interval $[0, T]$ for all $\nu \in \mathcal{V}$. This implies $a^i(0) \leq E[a^i(t_n)]$ for all $n \leq N$ and allows bounds on negative wealth to be described by portfolios that are self-financing or that allow withdrawals of consumption.[13]*

Finally, as in Section 4, we assume an equilibrium exists. The features important for our purposes are: the consumption market clears so that $\sum_{i=1}^I c^i(t_n) = \sum_{i=1}^I e^i(t_n) + \bar{\pi}_S D(t_n)$ at each consumption date t_n, and the asset markets clear so that $\sum_{i=1}^I W^i(t_n) = \bar{\pi}_S S^{\mathrm{ex}}(t_n)$ at each t_n. Market clearing and our assumption about the monotonicity of preferences imply $\bar{\pi}_S S^{\mathrm{ex}}(T) = 0$.

5.2   Bubbles and Bounds on Net Consumption

As in Section 3, the local martingale property of asset prices implies the inequality

$$S_k^{\mathrm{ex}}(0) \geq \sup_V E\left[\sum_{j=1}^{N+1} \rho\ (t_j) D_k(t_j) \right]$$

---

[12]Section 3 uses an assumption that some investor has regular preferences to derive this property from optimal choice. A similar result would hold in the multiperiod model given specific assumptions about investors' preferences in the multiperiod model.

[13]The latter property might be useful for constraints that allow investors to borrow up to the greatest lower bound of the present value of their future endowments when markets are incomplete.

for every asset k. If the inequality is strict, an asset's price exceeds the cost of superreplicating its dividends given nonnonegative wealth (we will show this later), and the asset's price will have a bubble according to Definition 3.1.

To rule out bubbles on the prices of the assets in positive net supply, we will use the following condition, and we will show it is automatically satisfied when the number of consumption dates is uniformly bounded. It is also satisfied under other assumptions sometimes used in the literature, as we will explain.

Condition 5.1. *There exists a nonnegative process γ such that every investor i's net consumption satisfies $c^i(t_n) - e^i(t_n) \leq \gamma(t_n)$, where*

$$\sup_V E \left[\sum_{n=1}^{N+1} \rho(t_n)\gamma(t_n) \right] < \infty. \tag{5.23}$$

The process γ in Condition 5.1 must be the same for every investor. We show in Appendix A.4 that Condition 5.1 implies there is a finite-cost portfolio with payouts that superreplicate the payout stream $\{\gamma_1, \ldots, \gamma_{N+1}\}$ and, consequently, that superreplicates the net consumption plans of any subset of investors. This portfolio maintains nonnegative wealth. We also show how to transform the results for the one-period model so that they also apply to the multiperiod model. Thus the same economic ideas presented there also apply here.

First, each investor who prefers more to less finances consumption net of endowments at its lowest possible cost, so each investor's financial wealth is bounded given Condition 5.1. Each investor i's equilibrium financial wealth must satisfy

$$W^i(t) \leq \operatorname{essup}_V E_t \left[\sum_{j=n+1}^{N+1} \frac{\rho(t_j)}{\rho(t)}\gamma(t_j) \right] \quad \text{on } [t_n, t_{n+1})$$

because, if this inequality did not hold, the investor would derive higher utility by appropriately switching to the superreplicating strategy for γ (see Lemma A.4) and consuming more than $\gamma + e^i$ and, consequently, more than the equilibrium consumption c^i. Second, this plus market clearing for the assets imply aggregate financial wealth must bounded: for all $n \leq N$,

$$\bar{\pi}_S S^{ex}(t) = \sum_{i=1}^I W^i(t) \leq I \times \operatorname{essup}_V E_t \left[\sum_{j=n+1}^{N+1} \frac{\rho(t_j)}{\rho(t)}\gamma(t_j) \right] \quad \text{on } [t_n, t_{n+1}). \tag{5.24}$$

This gives us the multiperiod analog of inequality (4.20), which was used in the one-period consumption model to rule out bubbles on the positive net supply assets. The following proposition provides the formal statement.

Proposition 5.1. *Suppose Condition 5.1 is satisfied. Then*

$$\bar{\pi}_S S^{ex}(0) = \sup_V E \left[\sum_{n=1}^{N+1} \rho(t_n)\bar{\pi}_S D(t_n) \right]. \tag{5.25}$$

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There is a $\nu \in \mathcal{V}$ such that

$$\bar{\pi}_S S^{ex}(0) = E\left[\sum_{n=1}^{N+1} \rho^*(t_n)\bar{\pi}_S D(t_n)\right]$$

and the process

$$\rho^*(t)\bar{\pi}_S S^{ex}(t) + \sum_{n=1}^{N+1} \rho^*(t_n)\bar{\pi}_S D(t_n)\mathbf{1}_{\{t_n \leq t\}},$$

is a martingale on $[0,T]$. Thus there are no bubbles on the price of any asset in positive net supply.

Proof. See Appendix A.4. □

Our main result is that there can be no bubbles on the prices of the positive net supply assets when the number of consumption dates is uniformly bounded. This is because every investor's net consumption is automatically bounded and Condition 5.1 is automatically satisfied for an appropriate choice of γ, as identified in our next result.

Theorem 5.1. *When the number of consumption dates is uniformly bounded by a finite number (i.e., when $N + 1 \leq \bar{n}$ for an integer $\bar{n} < \infty$), there are no bubbles on equilibrium price of any asset in positive net supply. In particular, Condition 5.1 is automatically satisfied by choosing γ to be*

$$\gamma(t_n) = \bar{\pi}_S D(t_n) + \bar{\pi}_S S^{ex}(t_n) - \sum_{i=1}^{I} a^i(t_n). \tag{5.26}$$

Proof. Given Proposition 5.1, the proof follows from the discussion in this section. □

The logic for the choice of γ is essentially the same as in the single-date consumption model. At each consumption date, the most an investor could consume is the aggregate dividend plus aggregate financial wealth *plus* the aggregate negative wealth permitted by the investors' lower bounds on wealth. This is the maximum amount of net consumption any investor could get through trading. The present value of this is finite at each consumption date. What is needed, however, in Condition 5.1 is value of the sum of the present values across consumption dates to be finite. This is automatic when the number of consumption dates has a uniform bound across states because

$$\sup_V E\left[\sum_{n=1}^{N+1} \rho(t_n)\left(\bar{\pi}_S D(t_n) + \bar{\pi}_S S^{ex}(t_n)\right) - \sum_{i=1}^{I} a^i(t_n)\right]$$

$$\leq (\bar{n} + 1)\bar{\pi}_S S(0) - \sum_{i=1}^{I} a^i(0) < \infty.$$

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As in the single-period model, this upper bound would prevent the accumulation of wealth needed for the existence of a bubble that affects aggregate financial wealth (i.e., the value of the positive net supply assets).

A necessary condition typically associated with the existence of bubbles on positive

We have shown that, under the assumption that investors prefer more to less, the finiteness of the present value of consumption net of endowments (the portion gotten through trade in the financial assets) determines whether or not bubbles can exist on the assets that determine aggregate financial wealth. We have shown that the present value of net consumption is automatically finite when the number of trading dates is uniformly bounded. This adds to the already known idea that infinitely many trade dates are necessary for bubbles on positive net supply assets.

5.3   Example

We now present an example of an equilibrium in which the price of a positive net supply asset has a bubble. The purpose of the example is to illustrate the importance of the number of consumption dates. In the example, the number of consumption dates is finite almost surely, but is not uniformly bounded across states. Otherwise, the critical economic quantities are uniformly bounded: The asset's price is uniformly bounded; therefore, so is the bubble on its price. The bubble has a finite lifespan. Consumption and private endowments are also uniformly bounded. All investors prefer more consumption to less, and choose the lowest cost portfolio to finance their net consumption. The lack of a uniform bound on the number of consumption dates causes Condition 5.1 to be violated, so consumption net of endowments cannot be superreplicated by a finite-cost portfolio in this example.

The financial market consists of two assets, one net unit of a "stock" that pays only a liquidating dividend of $3/2$ at date T and zero net units of a locally riskless bond. Their prices are S and B, and continuous trade is permitted over the deterministic time interval $[0, T]$. Uncertainty is described by two standard and independent Brownian motion processes Z_1 and Z_2.

The consumption dates constructed from a sequence of stopping times we now define. Define an exponential local martingale η by

$$\eta(t) = \exp\left(-\frac{1}{2} \int_0^t \psi^2(s)ds + \int_0^t \psi(s)dZ_1(s) \right),$$

where ψ is some given deterministic process having the properties

$$(\forall t \in [0, T)) \quad \int_0^t \psi^2(s)ds < \infty \quad \text{and} \quad \int_0^T \psi^2(t)dt = \infty$$

almost surely. Note η is independent of Z_2. The Novikov condition ensures $E[\eta(t)] = 1$ for all $t \in [0, T)$; however, $\eta(T) = 0$ almost surely and η is not a martingale. Define the stopping time τ by

$$\tau = \inf\left\{ t \in [0, T] : \eta(t) = \frac{1}{2} \right\}$$

Notice $P(\tau < T) = 1$ since η is continuous and $\eta(T) = 0$. Now let $\{\sigma_i\}$ be any increasing sequence of stopping times dependent solely on Z_2 (independent of Z_1)

with the properties

$$(\forall i = 1, \ldots, \infty)\ (\forall t \in [0, T])\ P(\sigma_i < t) > 0 \text{ and } P(\sigma_i < T) \downarrow 0 \text{ as } i \to \infty. \quad (5.28)$$

Let N be the random number defined by $\inf\{i : \sigma_i < \tau\} + 1$ if the infimum exists or by zero otherwise. The properties in (5.28) ensure N is well-defined, satisfies $N \geq 2$, and is finite almost surely but not uniformly bounded. The consumption dates along a given path ω in our example are $t_1(\omega) = \sigma_1(\omega) \wedge \tau(\omega), \ldots, t_i(\omega) = \sigma_i(\omega) \wedge \tau(\omega), \ldots,$ $t_{N()}(\omega) \equiv \tau(\omega)$, $t_{N()+1}(\omega) \equiv T$.

We use these consumption dates as the basis of a continuous-time overlapping generations model (Samuelson, 1958). A single representative investor represents each generation. The first, "generation 0," is endowed with one share of stock and participates in the financial market until the stopping time t_1, when it must consume from its financial wealth and depart from the economy. Any subsequent generation $i \geq 1$ arrives with the endowment

$$e_i = \frac{1}{2} + \frac{1}{2\eta(t_i)}.$$

and may trade until t_{i+1}, at which time it must consume from its financial wealth and depart from the economy. This process repeats with generation $i + 1$ until $t_{i+1} = \tau$. The generation arriving at τ is the last, receives the endowment $e = \frac{1}{2} + \frac{1}{2()} = \frac{3}{2}$, and consumes from its financial wealth at time T. The economy then ends. No generation may participate in the economy before the generation arrives or after it departs. The number of generations born along a given path ω is $N(\omega) < \infty$, and the total number of consumption dates is $N + 1$. Our construction ensures $N + 1$, the number of consumption dates and the number of generations, is finite almost surely but is not bounded by a constant.

Here is generation i's choice problem.

Choice Problem 5.2 (Generation i's Choice Problem). *On the time interval $[t_i, t_{i+1}]$, generation i chooses a portfolio (π_B^i, π_S^i) to maximize its expected utility*

$$E_{t_i}\left[\log\left(c^i(t_{i+1}) - \frac{1}{2}\right)\right],$$

subject to

$$W^i(t_i) = e_i, \quad dW^i(t) = \pi_B^i(t)dB(t) + \pi_S^i(t)dS(t), \quad W^i(t_{i+1}) \geq c^i(t_{i+1})$$

and $(\forall t \in [0, T])\ W^i(t) \geq 0$, P-almost surely. Expected utility equals minus infinity if $P\left(c^i(t_{i+1}) \leq \frac{1}{2}\right) > 0$

the spirit of "safety-first rules" (Roy, 1952).[14] Any generation born at time $t_i < \tau$ is uncertain whether it is the next to last generation, but generation τ knows it will be the last. In our example, no generation can hedge its departure time using the financial assets. The nonnegative wealth constraint serves to make doubling strategies infeasible; our results would be the same if negative wealth were bounded by any exogenous or endogenous negative number.

We prove below that equilibrium price system for this example is

$$S(t) = \frac{1}{2} + \frac{1}{2\eta(t \wedge \tau)} \quad \text{and} \quad B(t) \equiv 1. \tag{5.29}$$

Note $S(t_i) = e_i$ for all $i \le N$ and $S(T) = \frac{3}{2}$. This equilibrium stock price exceeds the lowest cost of replicating its terminal (and only) dividend of 3/2. Thus the stock has a bubble according to Definition 3.1, even though it is in positive net supply.[15] To see this, note that the initial stock price is 1. But with initial investment of 3/4, borrowing 3/4 at the locally riskless rate and buying 3/2 units of the stock also pays 3/2 at time T while maintaining nonnegative wealth of $3/(4\eta(t))$. No generation $i < N$ switches to the cheaper strategy because of the risk of needing to liquidate its portfolio prior to time τ (switching would yield negative infinite expected utility).[16]

Proposition 5.1 implies Condition 5.1 is violated. Consistent with Proposition 4.1, each generation i finances its consumption $c^i(t_{i+1}) = S(t_{i+1})$ at the lowest possible cost. Our proof below shows there is a ν for which $\rho^*(t)S(t)$ is a martingale during the generation i's lifetime $[t_i, t_{i+1}]$, yet for all $\nu \in \mathcal{V}$ the process $\rho(t)S(t)$ is strictly a nonnegative local martingale (a supermartingale) over the intervals $[t_i, \tau]$ and $[t_i, T]$ when $i < N$. Thus a generation born at time $t_i < \tau$ that would happen to know for certain it would survive until τ would not optimally hold the stock, but no generation in our example has this knowledge.

The remainder of this section proves the equilibrium for the example.

Proof of the Equilibrium: Our candidate equilibrium prices are (5.29), and the candidate equilibrium strategy of each generation consists of buying and holding the stock and consuming its financial wealth. Clearly all markets clear given these strategies, so the remaining issue is whether the strategies maximize expected utility given the candidate equilibrium prices.

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[14]The preferences also reflect aspects of goal-setting for intolerance for declines in standard of living (Dybvig, 1995), portfolio insurance (Leland, 1980; Grossman and Zhou, 1996), life-cycle concerns (Mariger, 1987), infinite risk-aversion (Campbell and Viceira, 2001), regulations requiring certain institutions to maintain liquid reserves, and mandated spending rules for university endowments (Dybvig, 1999).

[15]There is also a bond bubble, but this is less interesting because the bond is in zero net supply.

[16]Consistent with Proposition 3.3, the ability to replicate the dividend at a lower cost implies $S(0) > \sup_{\nu \in \mathcal{V}} E[\nu(T)S(T)]$. To see this, first note that $\rho^0(t) = (t \wedge)$, and that each ν has the form

$$\rho^\nu(t) = \rho^0(t) \exp\left(-\frac{1}{2}\int_0^t {}^2(s)ds - \int_0^t (s)dZ_2(s)\right). \tag{5.30}$$

In particular, $\rho^0(T) = 1/2$, so $E[\rho^\nu(T)] \le 1/2$. Moreover, $1 = S(0) > \frac{3}{4} \ge E[\rho^\nu(T)S(T)]$ for all $\in \mathcal{V}$, which implies inequality (5.30).

We prove this for a given generation i that arrives at time $t_i < T$ and departs at time $t_{i+1} \leq T$. The proof has two steps. The first step shows that $S(t_i)$ is the lowest cost of obtaining the payout of $S(t_{i+1})$, which is necessary for the candidate strategy of buying and holding the stock to be optimal. The second step shows $S(t_{i+1})$ provides the highest expected utility for generation i given its budget constraint.

To perform the first step, we show there is a $\nu \in \mathcal{V}$ such that

$$E_{t_i}\left[\frac{\rho^*(t_{i+1})}{\rho^*(t_i)}S(t_{i+1})\right] = S(t_i),$$

(see Proposition 3.3). Recall that $t_i = \sigma_i \wedge \tau < T$ by construction, and that τ and σ_{i+1} are independent. In the equilibrium, $\rho^0(t) = \eta(t)$, which is independent of σ_{i+1}. Now define the process M by

$$M(t) = \frac{P_t(\sigma_{i+1} < T)}{P(\sigma_{i+1} < T)},$$

where P_t denotes the time-t conditional probability. This M has the following properties: it is a bounded martingale with $M(t) > 0$ for $t \in [0, T)$, its terminal value $M(T)$ is either zero or $1/P(\sigma_{i+1} < T)$, and $M(t)$ is independent of both $S(t)$ and $\rho^0(t)$ at any time $t \in [0, T]$. These properties imply the process $M\rho^0$ is a nonnegative local martingale and strictly positive prior to time T. It also follows that $M\rho^0 S$ is a nonnegative local martingale. By the Martingale Representation Theorem, there is a $\nu \in \mathcal{V}$ such that $\rho^*(t) = M(t)\rho^0(t)$ on the random interval $[0, \tau]$ (Protter, 1992, Theorem IV.3.42). Moreover, direct computation shows

$$
\begin{aligned}
E[\rho^*(t_{i+1})S(t_{i+1})] &= E[M(t_{i+1})\rho^0(t_{i+1})S(t_{i+1})] \\
&= E[M(\sigma_{i+1} \wedge T)\rho^0(t_{i+1})S(t_{i+1})] = E[M(T)\rho^0(t_{i+1})S(t_{i+1})] \\
&= \frac{1}{P(\sigma_{i+1} < T)}E[\rho^0(t_{i+1})S(t_{i+1})1_{\{\sigma_{i+1}<T\}}] \\
&= \frac{1}{P(\sigma_{i+1} < T)}\left\{\frac{1}{2}P(\sigma_{i+1} < T) + \frac{1}{2}E[\eta(t_{i+1})1_{\{\sigma_{i+1}<T\}}]\right\} \\
&= \frac{1}{P(\sigma_{i+1} < T)}\left\{\frac{1}{2}P(\sigma_{i+1} < T) + \frac{1}{2}\int_0^T E[\eta(t \wedge \tau)]P(\sigma_{i+1} \in dt)dt\right\} = 1.
\end{aligned}
$$

Since $\rho^*(0)S(0) = 1$, this shows ρ^*S is a martingale on the interval $[0, t_{i+1}]$, so it is also a martingale on generation i's lifetime $[t_i, t_{i+1}]$. By Proposition 3.3, there is no feasible trading strategy that provides a higher payo than $S(t_{i+1})$ at a lower cost than $S(t_i)$. We remark that both ρ^*B and henc227-1.793Td[(i)]TFi1130Td[(T)32

that satisfies $W(t_{i+1}) \geq 1/2$, we have $W(t) \geq 1/2$ for $t \in [t_i, t_{i+1}]$.[17] It follows that for any feasible trading strategy,

$$E_{t_i}\left[\rho^0(t_{i+1})\left(W(t_{i+1}) - \frac{1}{2}\right)\right] \leq \rho^0(t_i)\left(e_i - \frac{1}{2}\right).$$

Direct calculation shows equality holds when $W(t_{i+1}) = S(t_{i+1})$. Given the strategy of buying and holding the stock, generation i's marginal utility is $u(S(t_{i+1})) = \rho^0(t_{i+1})/2$. Thus

$$E[u(S(t_{i+1}))] - E[u(W(t_{i+1}))]$$
$$\geq E\left[u(S(t_{i+1}))\left(S(t_{i+1}) - \frac{1}{2} - \left(W(t_{i+1}) - \frac{1}{2}\right)\right)\right] \geq 0,$$

so the equilibrium strategy does indeed maximize generation i's utility. $\qquad\square$

6 Equilibrium Equivalent Martingale Measures

This section presents new results about the existence of equivalent martingale measures. The results presented here follow directly from our preceding analysis of bubbles on positive net supply assets. They clarify the economics of the di erent partial equilibrium conclusions in the literature about pricing using equivalent martingale measures, and they identify new restrictions that would be implied by the added requirement of market clearing in an equilibrium. The results might be of independent

2. each $a^i Y$ $(i = 1, \ldots, I)$ is a Q^Y-martingale.

Inequality (6.31) would be implied, for example, by the assumption that some positive net supply asset pays a strictly positive dividend at maturity.

Proof. Theorem 4.1 implies there is a $\nu \in \mathcal{V}$ such that $\rho^* \left(\bar{\pi} S - \sum_{i=1}^{I} a^i \right)$ is a martingale. This martingale is strictly positive given (6.31). Define the new numeraire

$$Y(t) = \frac{1}{\bar{\pi} S(t) - \sum_{i=1}^{I} a^i(t)}.$$

Define the redominated values $\tilde{B}(t) = B(t)Y(t)$, $\tilde{S}(t) = S(t)Y(t)$, and $\tilde{\rho}(t) \equiv \rho(t)\frac{Y(0)}{Y(t)}$. By construction, $\tilde{\rho}^*(T)\tilde{B}(T)$ is strictly positive and $E[\tilde{\rho}^*(T)\tilde{B}(T)] = 1$, so we can define equivalent probability measure Q^Y by

$$(\forall A \in \mathcal{F}_T) \quad Q^Y(A) = E[\tilde{\rho}^*(T)\tilde{B}(T)\mathbf{1}_A].$$

That $S^k Y$ and $a^i Y$ are Q^Y-martingales follows from Theorem 4.1 and Bayes' Rule (Karatzas and Shreve, 1988, Lemma 3.5.3). $\qquad\square$

Proposition 6.1 provides conditions for the equilibrium prices of the positive net supply assets to be described as

$$S_k(0) = E^{Q^Y} \left[Y(T)S_k(T) \right],$$

and indicates equilibrium constraints on negative wealth satisfy

$$a^i(0) = E^{Q^Y} \left[Y(T)a^i(T) \right].$$

Studies of equivalent martingale measures typically use the bond price B as the numeraire (e.g., Harrison and Kreps (1979), Dybvig and Huang (1988), Delbaen and Schachermayer (1994, 1995, 1997a,b), and Loewenstein and Willard (2000a)). This would say asset prices are equal to the risk-neutral value of their payouts discounted at the locally riskless rate. The choice of B as the numeraire is appropriate if there is a $\nu \in \mathcal{V}$ such that ρB is a martingale; however, this feature is not automatic because the bond is in zero net supply and, as such, its price might have a bubble. But a typical assumption about the constraints on negative wealth allow B to be the numeraire, as our next result explains.

Corollary 6.1. *Assume an equilibrium exists when every investor prefers more terminal wealth to less and some investor has regular preferences and positive initial wealth. Given our standing assumption that B is strictly positive, if*

1. *at least one investor faces an exogenous constraint on negative wealth a^i in Problem 2.1 satisfying $a^i(t) \leq -\gamma B(t)$ where $\gamma > 0$, or*

2. *at least one investor has an endogenous constraint set $\mathcal{A}^i$ in Problem 2.2 for which there exists $a^i \in \mathcal{A}^i$ satisfying $a^i(t) \leq -\gamma B(t)$ where $\gamma > 0$,*

then there is an equivalent probability measure Q^B for which S^k/B is a martingale for each asset k in positive net supply.

The assumptions about the negative wealth constraints in Corollary 6.1 allow sustained short sales of the bond, which would permit an investor to arbitrage a bond bubble. Hence no bond bubble can exist under the assumptions, and the bond price B can serve as a numeraire for an equivalent martingale measure.

Corollary 6.1 provides an *equilibrium* restriction that is much stronger than the usual *partial equilibrium* "equivalent local martingale measure" (ELMM) as defined by Delbaen and Schachermayer (1995). An ELMM requires the martingale property for B, but allows the remaining discounted prices to be Q^B-local martingales. In effect, the partial equilibrium construction of the ELMM allows unconstrained and sustained short sales of the numeraire (so the bond price must be a Q^B martingale) but restricts (by the wealth constraint) an investor's ability to arbitrage bubbles on the remaining asset prices. Our result shows that market clearing imposes the additional restriction that the discounted equilibrium prices of positive net supply assets must nonetheless be Q^B-martingales. The use of Q^B for pricing depends on the choice of B as the numeraire for the wealth constraints, and does not identify other possible restrictions that reflect other reasonable numeraires for the constraints.

A Appendix: Proofs

A.1 Preliminary Lemma

Several of our proofs use the following lemma regarding the local martingale property of wealth and constraints on negative wealth.

Lemma A.1. *Let θ be a self-financing trading strategy and W its wealth process. Then ξW is a local martingale for every $\xi \in \mathcal{V}$. Consequently, for any bound on negative wealth a satisfying Assumption 2.1, ξa is a local martingale for every $\xi \in \mathcal{V}$.*

Proof. Let $\xi \in \mathcal{V}$ be given. Ito's Lemma implies

$$\xi(t)W(t) = w + \int_0^t \xi(s)\Big(\tilde{\theta}_S(s)\sigma(s) - W(s)\big(\kappa(s) + \phi(s)\big)\Big)dZ(s), \qquad \text{(A.1)}$$

where the K-dimensional row vector $\tilde{\theta}_S$ represents the dollar investments in the K risky assets (i.e., $\tilde{\theta}(t) = (\theta_k(t)S_k(t))_{k=1,\ldots K}$). (For this calculation, see, for example, Loewenstein and Willard (2000b).) The integrand in (A.1) is locally bounded, so ξW is a local martingale (Karatzas and Shreve, 1988, Chapter 3). That ξa is a local martingale follows from a's being the value of a self-financing portfolio, as Assumption 2.1 requires. $\square$

A.2 Proofs for Section 3.1

Proof of Proposition 3.1. We use proof by contradiction. Suppose no $\tilde{\eta}$ solves (3.7). Then Karatzas and Shreve (1998, Theorem 1.4.2) show there is a self-financing portfolio that requires no investment, provides a payout that is nonnegative and positive with positive

probability, and maintains pathwise nonnegative wealth. This would be a traditional "arbitrage opportunity" that could be undertaken as a net trade at any scale starting from any feasible consumption plan, the existence of which is contradict the existence of optimal choice for an investor who prefers more to less and solves Problem 2.1 or Problem 2.2. □

We first prove in Lemma A.2 that $P(\pi^0(T) = 0) > 0$ implies the existence of an approximate arbitrage. Then we prove Proposition 3.2.

Lemma A.2. *Suppose $P(\pi^0(T) = 0) > 0$, and let $\{\delta_n\}$ and $\{\gamma_n\}$ be sequences with $\delta_n \downarrow 0$ and $\gamma_n \uparrow \infty$. Then there exists a sequence of nonnegative random variables Y_n and a sequence of self-financing strategies θ^n such that*

1. *the initial investment is $w_n = \delta_n$,*
2. *the payoff on $\{\pi^0(T) = 0\}$ is $W^n(T) \geq \gamma_n Y_n$ with $\lim_n Y_n = Y$ almost surely, where $P(Y \geq 0) = 1$ and $P(Y > 0) > 0$, and*
3. *the corresponding wealth process W^n is pathwise nonnegative.*

Loewenstein and Willard (2000a) call approximate arbitrages like these "cheap thrills." Unlike theirs, our approximate arbitrage is robust to incomplete markets.

Proof. Recall $\tau = \inf\{t \in [0,\infty) \mid \pi^0(t) = 0\} \wedge T$, and define the increasing sequence of stopping times $\{\tau_n = \inf\{t \in [0,\infty) \mid \pi^0(t) = \frac{\delta_n}{\gamma_n}\} \wedge T$, where $\wedge$ denotes minimum. Note $\tau_n \to \tau$ almost surely. Consider the wealth process

$$W^n(t) = \begin{cases} \dfrac{\delta_n}{\pi^0(t)} & t \in [0, \tau_n] \\[2mm] \gamma_n \exp\left(\int_{\tau_n}^t r(s)\,ds\right) & t \in (\tau_n, T] \end{cases} \tag{A.2}$$

Note W^n is pathwise nonnegative and continuous, and $\pi^0(t)W^n(t) = \delta_n$ on $[0, \tau_n]$. Using (A.1) with $\equiv 0$, we see that W^n is the wealth process corresponding to the initial investment $w_n = \delta_n$ and the following investment strategy: on $[0, \tau_n]$ invest nothing in the bond ($\theta_B^n(t) \equiv 0$) and the dollar amount $\tilde{\theta}_S(t)$ found by setting $\tilde{\theta}_S(t)/W^n(t)$ equal to the orthogonal projection of (t) onto the range of (t) (this is the optimal growth portfolio); followed on $(\tau_n, T]$ by investing $W^n(\tau_n)$ in the bond until maturity ($\theta_B^n(t) \equiv \gamma_n$) and $\theta_S^n(t) \equiv 0$. This strategy is progressively measurable (Karatzas and Shreve, 1998, Lemma 1.4.4). Because $W(\tau_n) = \gamma_n$ when $\tau_n < T$, the strategy θ^n pays $W^n(T) = \gamma_n \exp(\int_{\tau_n}^T r(s)\,ds)$ on $\{\pi(T) = 0\}$ and $W^n(T) = \frac{\delta_n}{\pi^0(T)} \geq 0$ on $\{\pi(T) > 0\}$. The proof follows from setting $Y_n = \exp(\int_{\tau_n}^T r(s)\,ds)\mathbf{1}_{\{\pi^0(T)=0\}}$ and $Y = \exp(\int_\tau^T r(s)\,ds)\mathbf{1}_{\{\pi^0(T)=0\}}$ and noting $Y_n \to Y$ a.s. and $P(Y > 0) > 0$. □

We now prove Proposition 3.2.

Proof of Proposition 3.2. Let $W(T)$ be a candidate optimal terminal wealth for an investor who has regular preferences and initial wealth $w > 0$, and let $\theta = (\theta_B, \theta_S)$ be the corresponding investment strategy and W its wealth process. Optimality requires $W(t) \geq a(t)$ pathwise, where a is an exogenous bound in Problem 2.1 or is some $a \in \mathcal{A}^i$ in Problem 2.2.

We use proof by contradiction. Suppose $P(\pi^0(T) = 0) > 0$. Let $\{\theta^n\}$, $\{Y_n\}$, and Y be the quantities constructed in Lemma A.2. The definition of regular preferences implies there are sequences $\delta_n \downarrow 0$ and $\gamma_n \uparrow \infty$ and an n^* such that

$$U^i\big((1 - \delta_{n^*})W(T) + \delta_{n^*}Y_{n^*}\big) > U^i(W(T)).$$

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Consider the new self-financing strategy $(1 - \theta_{n*})\tilde{} + \tilde{}^{n*}$, which still requires initial wealth w but pays $(1 - \theta_{n*})W(T) + W^{n*}(T)$ (recall $\tilde{}$ denotes the strategy denominated in dollars). Note the new wealth process $(1 - \theta_{n*})W(t) + W^{n*}(t)$ pathwise exceeds $a(t)$. Because the investor also prefers more terminal wealth to less,

$$U^i\big((1 - \theta_{n*})W(T) + W_{n*}(T)\big) \geq U^i\big((1 - \theta_{n*})W(T) + \theta_{n*}Y_{n*}\big) > U^i(W(T)).$$

Thus $W(T)$ is not optimal for Problem 2.1 or 2.2, so optimality requires $P(\,^0(T) > 0) = 1$.

That $P(\,^0(T) > 0) = 1$ implies $(\forall \,\in \mathcal{V})\, P(\,(T) > 0) = 1$ follows our assumption in Section 2 that $\int_0^T \|\,(s)\|^2 ds < \infty$ and Revuz and Yor (1994, Exercise IV.3.25). $\qquad\square$

Proof of Proposition 3.3. (Statement 1:) Let $\,\in \mathcal{V}$ be given. Because a and W are both local martingales (Lemma A.1), their difference is a local martingale. By definition, there is an increasing sequence of stopping times $\{\,_n\} \uparrow T$ a.s. such that $W(0) - a(0) = E\big[\,(\,_n)(W(\,_n) - a(\,_n))\big]$ for all n (see Footnote 7). Each $\,(\,_n)(W(\,_n) - a(\,_n))$ is non-negative, so Fatou's Lemma implies

$$E\big[\,(T)W(T) - \,(T)a(T)\big] \leq \lim_n \big\{E\big[\,(\,_n)W(\,_n) - \,(\,_n)a(\,_n)\big]\big\} = w - a(0),$$

which implies (3.9).

(Statement 2:) Define $\hat{W}(t) = \operatorname*{essup}_{\,v} E_t\big[\frac{\nu(T)}{\nu(t)}(X - a(T))\big]$. For any $\,\in \mathcal{V}$, there is a modification of $\,\hat{W}$ that is a RCLL supermartingale, which we continue to denote by $\,(t)\hat{W}(t)$ (the proof is virtually identical to Karatzas and Shreve (1998, Theorem 5.6.5)). The Doob-Meyer decomposition and the Martingale Representation Theorem imply

$$\,(t)\hat{W}(t) = \hat{W}(0) + \int$$

the constraint a. We first construct a self-financing strategy which invests the withdrawals in the bond and has a payoff of at least $X - a(T)$. Define the bond trading strategy (in dollars) as $\tilde{\pi}_B(t) = \hat{W}(t) - \tilde{\pi}_S(t) + B(t) \int_0^t (1/B(s))\,d\hat{C}(s)$. Then the self-financing trading strategy $\tilde{\pi} = (\tilde{\pi}_B, \tilde{\pi}_S)$ has the wealth process $\tilde{W} = \tilde{\pi}_B + \tilde{\pi}_S$ that requires initial wealth $\tilde{W}(0) = \hat{W}(0) = w - a(0)$ and pays $\tilde{W}(T) \geq \hat{W}(T) = X - a(T)$ at maturity.

We now modify the wealth process to take advantage of negative wealth possibly permitted by a. Define $W(t) \equiv \tilde{W}(t) + a(t)$. Then $W(0) = \tilde{W}(0) + a(0) = w$ and $W(T) \geq X$. Because a corresponds to the value of a self-financing trading strategy α (see Assumption 2.1) and $\tilde{W}$ corresponds to a self-financing trading strategy $\tilde{\pi}$ defined above, it follows W corresponds to the self-financing trading strategy $\tilde{\pi} + \alpha$, and this strategy superreplicates the payoff X starting with initial wealth w. Moreover, since $\tilde{W}(t) \geq \hat{W}(t) \geq 0$, the final wealth process satisfies the bound on negative wealth $W(t) \geq a(t)$.

We now prove our earlier claim there is a progressively measurable trading strategy $\tilde{\pi}_S$ satisfying $\tilde{\pi}_S(t)\,\sigma(t) = \theta(t)$, using arguments similar to those of Karatzas and Shreve (1998, page 219). We first note (A.5) implies

$$\int_{(0,t]} \frac{dA(s)}{\delta(s)} = \hat{C}(t) + \int_0^t \theta(s)\,\lambda(s)\,ds \geq 0 \tag{A.7}$$

for all $t \in [0, T]$ because A is nondecreasing. We show $\theta(t)$ is in the range of $\sigma(t)$ for all $t \in [0, T]$. To this end, we first choose $\hat{\theta}(t)$ to be the process defined by the orthogonal projection of $\theta(t)$ onto the null space of $\sigma(t)$ at each time $t \in [0, T]$. The process $\hat{\theta}$

A.3 Proofs for Section 4

Proof of Proposition 4.1. The process $\nu(W^i - a^i)$ is a local martingale for every $\nu \in \mathcal{V}$ (see Lemma A.1). The process is also nonnegative so it is a supermartingale. Thus

$$W^i(t) - a^i(t) \geq \operatorname{essup}_{\nu \in \mathcal{V}} E_t\left[\frac{\nu(T)}{\nu(t)}(W^i(T) - a^i(T))\right]. \tag{A.8}$$

We now argue by contradiction: Suppose in an equilibrium an investor i who prefers more terminal wealth to less chooses a trading strategy so that inequality (A.8) is strict with positive probability given the equilibrium bound on negative wealth a^i. Then this investor's equilibrium wealth would satisfy

$$w^i - a^i(0) \geq \sup_{\hat{\nu} \in \mathcal{V}} E[\hat{\nu}(t)(W^i(t) - a^i(t))]$$

$$> \sup_{\hat{\nu} \in \mathcal{V}} E\left[\hat{\nu}(t) \operatorname{essup}_{\bar{\nu} \in \mathcal{V}} E_t\left[\frac{\bar{\nu}(T)}{\bar{\nu}(t)}(W^i(T) - a^i(T))\right]\right]$$

$$\geq E\left[\nu(t) E_t\left[\frac{\nu(T)}{\nu(t)}(W^i(T) - a^i(T))\right]\right] = E[\nu(T)(W^i(T) - a^i(T))]$$

for all $\nu \in \mathcal{V}$. This would imply $w^i - a^i(0) > \sup_{\hat{\nu} \in \mathcal{V}} E[\hat{\nu}(T)(W^i(T) - a^i(T))]$. Proposition 3.3 then says investor i could superreplicate his equilibrium terminal wealth $W^i(T)$ at a lower cost $\hat{w}$ and honor the wealth bound a^i (so the new strategy would be feasible for Problem 2.1 or Problem 2.2). By following this superreplicating strategy and investing the cost difference in the locally riskless bond, investor i would obtain higher utility. This contradicts the optimality of investor i's strategy required by an equilibrium. Thus (4.18) must hold in an equilibrium for every investor who prefers more terminal wealth to less. $\square$

Proof of Lemma 4.1. We first show the assumptions of the lemma imply

$$X(\tau) = \operatorname{essup}_{\nu} E\left[\frac{\nu(T)}{\nu(\tau)}X(T)\right],$$

P-almost surely for all stopping times τ. Proposition 3.3 will then imply $X(0)$ is the lowest cost of replicating $X(T)$ with a portfolio that maintains nonnegative wealth. Lemma A.3 (presented next) will prove the existence of a $\nu^* \in \mathcal{V}$ that makes $\nu^*(t)X(t)$ a martingale.

The process $V^\Lambda(t) = \operatorname{essup}_{\nu} E_t\left[\frac{\nu(T)}{\nu(t)}\right]$ exists and has an RCLL modification for which $\nu(t)V^\Lambda(t)$ is an RCLL supermartingale for any $\nu \in \mathcal{V}$ (the proof is virtually identical to Karatzas and Shreve (1998, Theorem 5.6.5)). Choose a fixed stopping time τ and an arbitrary $\hat{\nu} \in \mathcal{V}$. The properties of the essential supremum (see Footnote 9) and the Monotone Convergence Theorem imply, for any $\epsilon > 0$, there is a $\nu \in \mathcal{V}$ such that

$$E\left[\hat{\nu}(\tau)\right.^{\epsilon}$$

The supermartingale property of V^Λ implies

$$V(\ ) \geq E\left[\frac{^\epsilon(T)}{^\epsilon(\ )}\, 1_{\{\ _n = T\}} + \frac{^\epsilon(\ _n)}{^\epsilon(\ )}\, V^\Lambda(\ _n) 1_{\{\ _n < T\}}\right],$$

so we have

$$> E\left[\,\hat{}(\ )\frac{^\epsilon(\ _n)}{^\epsilon(\ )} V^\Lambda(\ _n) 1_{\{\ _n < T\}}\right].$$

Since $V^\Lambda \geq X$,

$$> E\left[\,\hat{}(\ )\frac{^\epsilon(\ _n)}{^\epsilon(\ )} X(\ _n) 1_{\{\ _n < T\}}\right].$$

Because $\ _n$ reduces the local martingale $\hat{}(\ )\frac{\nu^\epsilon(t)}{\nu^\epsilon(\ )} X(t)$ on $t \geq\ $, we have

$$E[\,\hat{}(\ )X(\ )] = E\left[\,\hat{}(\ )\frac{^\epsilon(T)}{^\epsilon(\ )} X(T) 1_{\{\ _n = T\}}\right] + E\left[\,\hat{}(\ )\frac{^\epsilon(\ _n)}{^\epsilon(\ )} X(\ _n) 1_{\{\ _n < T\}}\right]$$

$$\leq E\left[\,\hat{}(\ )\frac{^\epsilon(T)}{^\epsilon(\ )} X(T)\right] +$$

$$\leq E\left[\,\hat{}(\ )\,\text{essup}\ _V E\left[\frac{(T)}{(\ )} X(T)\right]\right] +\ .$$

Letting $\ \to 0$ we find

$$E[\,\hat{}(\ )X(\ )] \leq E\left[\,\hat{}(\ )\,\text{essup}\ _V\ \text{ssp}\quad w\right.$$

where each $\xi^m \in \mathcal{V}$ and $\{\tau^m\}$ is an increasing sequence of stopping times with $\tau^0 = 0$ and $\tau^m \uparrow T$ almost surely. The important properties of ξ^* are that, for all m,

$$E\left[\xi^*(\tau^m)X(\tau^m)\right] = X(0) \tag{A.11}$$

and

$$E\left[\xi^*(\tau^m)X(\tau^m)1_{\{\tau^m<T\}}\right] \leq \varepsilon_m \tag{A.12}$$

for a sequence $\{\varepsilon_m\}$ of real numbers that decrease to zero as $m \to \infty$. Property (A.11) implies ξ^*X is a nonnegative local martingale and, consequently, a supermartingale. Property (A.12) and the Monotone Convergence Theorem additionally imply

$$\lim_m E\left[\xi^*(T)X(T)1_{\{\tau^m=T\}}\right] = E\left[\xi^*(T)X(T)\right] = X(0),$$

so ξ^*X must have constant expectation and is therefore a martingale. We show later in the proof the ξ^* we construct is in $\mathcal{V}$.

We now construct a ξ^* of the form (A.10) using induction. Take as given a sequence of positive real numbers $\{\varepsilon_m\}$ for which $\varepsilon_m \downarrow 0$. We start by defining $\xi^1 \in \mathcal{V}$ and the stopping time τ^1. Using the definition of supremum (applied to (A.9) evaluated at $\tau = 0$), there is a $\xi^1 \in \mathcal{V}$ such that

$$E[\xi^1(T)X(T)] + \frac{1}{2} \geq X(0). \tag{A.13}$$

To find τ^1, we first define the increasing sequence of stopping times by $\tau^1_n = \inf\left\{t \mid \xi^1(t)X(t) \geq n\right\} \wedge T$. Because ξ^1X is a nonnegative local martingale (supermartingale), Doob's Maximal Inequality implies

$$P\{\tau^1_n < T\} = P\left\{\sup_{\{t\ [0,T]\}} \xi^1(t)X(t) \geq n\right\} \leq \frac{X(0)}{n}$$

(Revuz and Yor, 1994, Theorem II.1.7). Consequently, $\tau^1_n \uparrow T$ almost surely, and the Monotone Convergence Theorem implies there is an n^1 satisfying both $P\{\tau^1_{n^1} < T\} \leq \frac{X(0)}{n^1} < \frac{1}{2}$ and

$$E\left[\xi^1(T)X(T)1_{\{\tau^1_{n^1}<T\}}\right] < \frac{1}{2}. \tag{A.14}$$

Define $\tau^1 \equiv \tau^1_{n^1}$, and let $\xi(t) = \xi^1(0) + \xi^1(t)1_{\{0<t\ \tau^1\}}$ on the set $\{0 \leq t \leq \tau^1\}$.

Before proceeding, we verify properties (A.11) and (A.12) for the ξ^1 and τ^1 that define ξ up to this point. Note the stopped process $\xi^1(t\wedge\tau^1)X(t\wedge\tau^1)$ is a uniformly bounded and nonnegative local martingale, so it is a martingale by Lebesgue's Dominated Convergence Theorem. Therefore $E\left[\xi^1(\tau^1)X(\tau^1)\right] = E\left[\xi^*(\tau^1)X(\tau^1)\right] = X(0)$, which verifies (A.11) for $m = 1$. Given this, we have

$$E\left[\xi^1(\tau^1)X(\tau^1)1_{\{\tau^1<T\}}\right] + E\left[\xi^1(T)X(T)1_{\{\tau^1=T\}}\right] = X(0). \tag{A.15}$$

Subtracting (A.15) from (A.13) we obtain the inequality

$$E\left[\xi^1(T)X(T)1_{\{\tau^1<T\}}\right] + \frac{1}{2} - E\left[\xi^1(\tau^1)X(\tau^1)1_{\{\tau^1<T\}}\right] \geq 0$$

Our choice of n^1 in (A.14) therefore implies

$$E\left[\xi^1(\tau^1)X(\tau^1)1_{\{\tau^1<T\}}\right] = E\left[\xi^*(\tau^1)X(\tau^1)1_{\{\tau^1<T\}}\right] < \varepsilon_1.$$

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Thus our construction of Λ up to this point also satisfies property (A.12) for $m = 1$.

We continue our construction of Λ using induction. Let $j \geq 2$ be an integer, and suppose we have a Λ of the form $\Lambda(t) = \Lambda^1(0) + \sum_{m=1}^{j-1} \Lambda^m(t)1_{\{\tau^{m-1} < t \leq \tau^m\}}$ on the set $\{0 \leq t \leq \tau^{j-1}\}$ for which the properties (A.11) and (A.12) hold for all m, $1 \leq m \leq j-1$, given the sequence $\{\tau^m\}$. We are assuming

$$X(\tau^j) = \operatorname{essup}_{\nu} E_{\tau^j}\left[\frac{\Lambda(T)}{\Lambda(\tau^j)}X(T)\right],$$

so (per Footnote 9) there is a $\Lambda^j \in \mathcal{V}$ so that

$$E\left[\Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(T)}{\Lambda^j(\tau^{j-1})}X(T)\right] + \frac{\epsilon_j}{2} \geq E\left[\Lambda^{j-1}(\tau^{j-1})X(\tau^{j-1})\right] = X(0). \qquad (A.16)$$

To select τ^j, we first define the increasing sequence of stopping times for $n > n^{j-1}$ by

$$\tau^j_n = \inf\left\{t \mid t \geq \tau^{j-1} \text{ and } \Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(t)}{\Lambda^j(\tau^{j-1})}X(t) \geq n\right\} \wedge T.$$

Doob's Maximal inequality again implies $\tau^j_n \uparrow T$ almost surely. This and the Monotone Convergence Theorem implies there is an n^j satisfying both $P\{\tau_{n^j} < T\} \leq \frac{X(0)}{n^j} < \frac{\epsilon_j}{2}$ and

$$E\left[\Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(T)}{\Lambda^j(\tau^{j-1})}X(T)1_{\{\tau^j_{n^j} < T\}}\right] < \frac{\epsilon_j}{2}. \qquad (A.17)$$

Define $\tau^j \equiv \tau^j_{n^j}$, and on the set $\{0 \leq t \leq \tau^j\}$ let $\Lambda(t) = \Lambda^1(0) + \sum_{m=1}^{j} \Lambda^m(t)1_{\{\tau^{m-1} < t \leq \tau^m\}}$.

We verify properties (A.11) and (A.12) for $m \leq j$. The stopped process $\Lambda^j(t \wedge \tau^j)X(t \wedge \tau^j)$ is a uniformly bounded nonnegative local martingale, so it is a martingale by Lebesgue's Dominated Convergence Theorem. This implies $E\left[\Lambda^j(\tau^j)X(\tau^j)\right] = E\left[\Lambda^*(\tau^j)X(\tau^j)\right] = X(0)$, which verifies property (A.11) for $m = j$. The martingale property of the stopped process and the fact that property (A.11) holds for $m = j-1$ imply

$$X(0) = E\left[\Lambda^{j-1}(\tau^{j-1})X(\tau^{j-1})\right] = E\left[\Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(\tau^j)}{\Lambda^j(\tau^{j-1})}X(\tau^j)\right]$$

$$= E\left[\Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(\tau^j)}{\Lambda^j(\tau^{j-1})}X(\tau^j)1_{\{\tau^j < T\}}\right] + E\left[\Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(T)}{\Lambda^j(\tau^{j-1})}X(T)1_{\{\tau^j = T\}}\right]$$
$$(A.18)$$

Subtracting (A.18) from (A.16) yields and applying (A.17), we obtain

$$E\left[\Lambda^*(\tau^j)X(\tau^j)1_{\{\tau^j < T\}}\right] = E\left[\Lambda^{j-1}(\tau^{j-1})\frac{\Lambda^j(\tau^j)}{\Lambda^j(\tau^{j-1})}X(\tau^j)1_{\{\tau^j < T\}}\right] < \epsilon_j,$$

which verifies property (A.12) for $m = j$.

Induction provides a sequence $\{\Lambda^m\}$ in $\mathcal{V}$ and an increasing sequence of stopping times $\{\tau^m\}$. Because $P(\tau^m < T) \downarrow 0$, the process $\Lambda(t) = \Lambda^1(0) + \sum_{m=1} \Lambda^m(t)1_{\{\tau^{m-1} < t \leq \tau^m\}}$ is defined on $[0, T]$. Its construction makes it progressively measurable. To verify $\Lambda \in \mathcal{V}$,

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we must check the integrability condition $\int_0^T \|\theta(t)\|^2 dt < \infty$ almost surely. Note that $\theta^*(T) = \theta^m(T)$ on the event $\{\tau_m = T\}$, and $P(\theta^m(T) > 0) = 1$ since $\theta_m \in \mathcal{V}$ and $P(\theta^0(T) > 0) = 1$ (see Proposition 3.2). Therefore $P(\theta^*(T) = 0) \leq P(\tau_m < T) \to 0$ as $m \to \infty$, so $P(\theta^*(T) = 0) = 0$. The integrability condition then follows from Revuz and Yor (1994, IV.3.25)

SUFFICIENCY: Because θX is a nonnegative local martingale for all $\theta \in \mathcal{V}$, it is also a supermartingale. Therefore $\sup_{\theta \in \mathcal{V}} E[\theta(T)X(T)] \leq X(0)$. If there exists a θ that makes $\theta^* X$ a martingale, then we have $X(0) = E[\theta^*(T)X(T)] \leq \sup_{\theta \in \mathcal{V}} E[\theta(T)X(T)] \leq X(0)$, which proves the statement. $\square$

A.4   Proofs for Section 5

To prove Proposition 5.1, we first state the following lemma.

Lemma A.4. *Consider a process π that satisfies Condition 5.1, and let $t \in [0, T]$ be given. Starting at time t, the lowest cost portfolio that superreplicates the payouts $\pi(t_n)$ for $t_n > t$ given pathwise nonnegative wealth is*

$$\underset{\theta \in \mathcal{V}}{\operatorname{essup}}\ E_t\left[\sum_{t_n > t} \frac{\theta(t_n)}{\theta(t)}\pi(t_n)\right] = \underset{\theta \in \mathcal{V}}{\operatorname{essup}}\ E_t\left[\frac{\theta(T)}{\theta(t)}\sum_{t_n > t}\frac{\theta^0(t_n)}{\theta^0(T)}\pi(t_n)\right].$$

Lemma A.4 says the time t "present value" of the payout stream $\{\pi(t_n) : t_n > t\}$ equals the lowest cost of a portfolio that reinvests each future payout in the optimal growth portfolio (represented by θ^0) and pays out the resulting amount at maturity. We will prove Lemma A.4 after we prove Proposition 5.1.

Proof of Proposition 5.1. Lemma A.4 allows us to convert the bound on $\bar{}_s S^{\mathrm{ex}}$ in (5.24) to one that conforms to the single-period bound used in Lemma 4.1. It follows from inequality (5.24) and Lemma A.4 that

$$\bar{}_s S^{\mathrm{ex}}(t) \leq I \times \underset{\theta \in \mathcal{V}}{\operatorname{essup}}\ E_t\left[\frac{\theta(T)}{\theta(t)}\sum_{t_n > t}\frac{\theta^0(t_n)}{\theta^0(T)}\pi(t_n)\right].$$

Letting $X(t) = \bar{}_s S^{\mathrm{ex}}(t) + \sum_{t_n \leq t}\frac{\theta^0(t_n)}{\theta^0(t)}\bar{}_s D(t_n)$ and $\pi = I \times \sum_{n=1}^{N+1}\frac{\theta^0(t_n)}{\theta^0(T)}\pi(t_n)$ in Lemma 4.1, we have

$$\bar{}_s S^{\mathrm{ex}}(t) + \sum_{t_n \leq t}\frac{\theta^0(t_n)}{\theta^0(t)}\bar{}_s D(t_n) = \underset{\theta \in \mathcal{V}}{\operatorname{essup}}\ E_t\left[\frac{\theta(T)}{\theta(t)}\sum_{n=1}^{N+1}\frac{\theta^0(t_n)}{\theta^0(T)}\bar{}_s D(t_n)\right].$$

Equation (5.25) follows by evaluating the preceding equality at time 0, and applying

Lemma A.4. Lemma 4.1 also implies there is a $\in \mathcal{V}$ such that

$$\bar{s}S^{\text{ex}}(0) = E\left[\ ^*(T)\sum_{n=1}^{N+1}\frac{\ ^0(t_n)}{0}\right.$$

is pathwise nonnegative. It also corresponds to the wealth process of a portfolio strategy that withdraws (t_n) at each date $t_n > t$ and shorts the dollar amount (t_n) in the optimal growth portfolio on the remaining interval $[t_n, T]$. As this is one way to generate the payouts $\{ (t_n) : t_n > t\}$, the lowest cost (t) (starting from time t) of generating the payouts $\{ (t_n) : t_n > t\}$ must satisfy

$$(t) \leq \text{essup}_v E_t\left[\frac{(T)}{(t)} \sum_{t_n > t} \frac{{}^0(t_n)}{{}^0(T)} (t_n)\right]$$

$$= \text{essup}_v E_t\left[\sum_{t_n > t} \frac{{}^0(t_n)}{(t)} (t_n) \exp\left(-\int_0^T (t)\,dZ(t) - \frac{1}{2}\int_0^T \| (t)\|^2 dt\right)\right]$$

$$= \text{essup}_v E_t\left[\sum_{t_n > t} \frac{(t_n)}{(t)} (t_n) \exp\left(-\int_{t_n}^T (t)\,dZ(t) - \frac{1}{2}\int_{t_n}^T \| (t)\|^2 dt\right)\right]$$

$$\leq \text{essup}_v E_t\left[\sum_{t_n > t} \frac{(t_n)}{(t)} (t_n)\right]. \quad \text{(A.22)}$$

On the other hand, for any portfolio strategy that maintains nonnegative wealth $W(s)$ has payouts $\hat{}(t_n) \geq (t_n)$ for $t_n > t$, the process $(s)W(s) + \sum_{t < t_n \ s} (t_n)\hat{}(t_n)$ is a nonnegative supermartingale. Thus, starting at time t, the lowest-cost superreplicating portfolio that maintains nonnegative wealth must satisfy

$$(t) \geq \text{essup}_v E_t\left[\sum_{t_n > t} \frac{(t_n)}{(t)} \hat{}(t_n)\right] \geq \text{essup}_v E_t\left[\sum_{t_n > t} \frac{(t_n)}{(t)} (t_n)\right]$$

$$\geq \text{essup}_v E_t\left[\frac{(T)}{(t)} \sum_{t_n > t} \frac{{}^0(t_n)}{{}^0(T)} (t_n)\right] \geq (t),$$

where the last two inequalities follow from (A.22). This completes the proof of the lemma. $\square$

References

J. Y. Campbell and L. M. Viceira. Who should buy long-term bonds? *American Economic Review*, 91:99–127, 2001.

Y. S. Chow and H. Teicher. *Probability Theory*. Springer-Verlag, New York, New York, 3 edition, 1997.

F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematiche Annalen*, 300:463–520, 1994.

F. Delbaen and W. Schachermayer. The existence of absolutely continuous local martingale measures. *The Annals of Applied Probability*, 5(4):926–945, 1995.

F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. mimeo, Institut für Statistik der Unversität Wien, 1997a.

F. Delbaen and W. Schachermayer. Non-arbitrage and the fundamental theorem of asset pricing: Summary of main results. *Proceedings of Symposia in Applied Mathematics*, 00: 1–10, 1997b.

F. Delbaen and W. Schachermayer. A simple counterexample to several problems in the theory of asset pricing. *Mathematical Finance*, 8(1):1–11, 1998.

P. H. Dybvig. Duesenberry's ratcheting of consumption: Optimal dynamic consumption and investment given intolerance for any decline in standard of living. *Review of Economic Studies*, 62:287–313, 1995.

P. H. Dybvig. Using asset allocation to protect spending. *Financial Analysts Journal*, pages 49–62, January–February 1999.

P. H. Dybvig and C. Huang. Nonnegative wealth, absence of arbitrage, and feasible consumption plans. *The Review of Financial Studies*, 1(4):377–401, 1988.

P. H. Dybvig and S. A. Ross. Arbitrage. In J. Eatwell, M. Milgate, and P. Newman, editors, *The New Palgrave: Finance*, pages 57–71. MacMillan Press, New York, 1987.

S. J. Grossman and Z. Zhou. Equilibrium analysis of portfolio insurance. *The Journal of Finance*, 51(4):1379–1403, 1996.

J. M. Harrison and D. M. Kreps. Speculative investor behavior in a stock market with heterogenous expectations. *Quarterly Journal of Economics*, 92(2):323–336, 1978.

J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–408, 1979.

J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications*, 11:215–260, 1981.

S. Heston, M. Loewenstein, and G. A. Willard. Options and bubbles. *Review of Financial Studies*, 20(2):359–390, March 2007.

I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York, 2 edition, 1988.

I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance*. Springer-Verlag, New York, 1998.

H. E. Leland. Who should buy portfolio insurance? *Journal of Finance*, 35(2):581–594, 1980.

M. Loewenstein and G. A. Willard. Local martingales, arbitrage, and viability: Free snacks and cheap thrills. *Economic Theory*, 16:135–161, 2000a.

M. Loewenstein and G. A. Willard. Rational equilibrium asset-pricing bubbles in continuous trading models. *Journal of Economic Theory*, 91(1):17–58, 2000b.

M. Magill and M. Quinzii. Infinite horizon incomplete markets. *Econometrica*, 62(4):853–880, 1994.

R. Mariger. A life-cycle consumption model with liquidity constraints: Theory and empirical results. *Econometrica*, 55:533–557, 1987.

P. E. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, New York, 1992.

D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer-Verlag, New York, 2 edition, 1994.

A. D. Roy. Safety first and the holding of assets. *Econometrica*, 20(3):432–449, 1952.

P. A. Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66(6):467–482, 1958.

M. S. Santos and M. Woodford. Rational asset pricing bubbles. *Econometrica*, 65(1):19–57, 1997.

J. Tirole. On the possibility of speculation under rational expectations. *Econometrica*, 50: 1163–1181, 1982.