Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVI[illegible]

JOURNAL OF
Economic
Theory

www.elsevier.com/locate/jet

[illegible]

Abstract

This pa[illegible]d: players evaluate t[illegible]e discounted stream of [illegible]ghted behavior of agents [illegible]ibrium process of network f[illegible]which no equilibrium strategy p[illegible]ditions under which the equilibriu[illegible]

JEL classi[illegible]

Keywords: [illegible]

1. Intro[illegible]

In sev[illegible]s, the structure of interactions between individuals is best d[illegible]precise structure of interaction across players may be cruci[illegible]ome. Examples include channels of information flow [3,6,7,22], trading networks [30,31,35], mutual insurance [15,16], technology adoption

* Corres[illegible]

E-mail [illegible].dutta@warwick.ac.uk (B. Dutta), s.ghosal@warwick.ac.uk (S. Ghosal), Debraj.ray[illegible]@nyu.edu (D. Ray).

0022-0531[illegible]ee front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.101[illegible]j.jet.2004.05.001

[9,4] and buyer–seller networks [24,25,32].[1] Most of these papers explicitly adopt the network formalism, and describe the space of interactions as a graph, where the set of nodes coincides with the set of agents, while an arc between two nodes indicates the existence of bilateral interaction between the corresponding agents.

The theoretical literature on networks, starting from [2,21], emphasizes two related issues. The first issue is the determination of the structure of networks which will be formed if links are established voluntarily by agents so as to maximize individual self-interest, while the second issue is concerned with whether such endogenous networks are socially efficient.

Following [2], the typical approach has been to model network formation in a static framework, aware of this occurrence. In contrast, we assume that a unilateral deviation by a player that breaks links other than the one with her partner cannot be used as a conditioning device by the partner. We are aware, of course, that this restriction is not entirely satisfactory—we rule out bilateral conditioning on unilateral action—but our own attempts to deal with both types of conditioning have led us into difficult terrain (concerning existence, even in mixed strategies) and we have settled for the more modest advance in this paper.

We show in Theorem 1 that a Markovian equilibrium process of network formation exists.

We use our solution concept to tackle the question of efficiency in networks. It is well known that "stable" networks may not be efficient, and the reason for this is simple. When a link is formed, or destroyed, the players involved do so with their own gain in mind. At the same time, these actions also affect the payoff of other players, and so a wedge is driven between stability and efficiency. Theorem 2 restates this in an explicitly dynamic context, using our solution concept: there are network structures where the process will not converge to any efficient network for *any* equilibrium strategy profile. This is the dynamic counterpart of the conflict between individual incentives and social efficiency demonstrated by [21] for static networks.

A simple way of seeing this conflict (at least for *some* equilibria) is to study network games where link formation is always profitable in the static sense (to the pair which forms the link). Call this property *link monotonicity*. Of course, when players i and j form an additional link, some player k may suffer a loss in current value. This implies that the complete network is not necessarily socially efficient. Nevertheless, Theorem 3 establishes that there is some equilibrium at which the complete graph is reached in the limit from all initial networks.

Yet other questions remain. For instance, how good is farsighted network formation at resolving "weaker" efficiency issues that stem, for instance, from nonconvexities or increasing returns? In particular, consider situations in which a "small" number of links are costly (to those who form them), while a larger number of links is beneficial to all. Jackson and Watts [20] observe that myopic agents cannot capture the benefits from such situations: the process may not get off the ground if initial returns are negative. No pair of agents may agree to form the first link if the immediate benefit is smaller than the cost, even if subsequent benefits are exceedingly large.

At first sight, it appears that farsightedness would automatically take care of this problem. A matched pair of agents would surely realize the future gains from linking, even if those benefits are not to be had in the short term. Yet this behavior applied across the board cannot constitute an equilibrium, for then a matched pair would prefer *not* to form a link until such time as a large number of links have already been built up. This would enable then to save on the transition costs when there are a small number of links. Just because agents are farsighted does not mean that they are impervious to short-term costs. Faced with a less costly transition path they would surely prefer such an alternative.

Notice that these efficiency issues are not as weak as coordination failures. There is some element of coordination, in that the efficient outcome is easy enough to sustain as an equilibrium, provided one starts there. But there is also a genuine absence of common interest: starting from the null network, for instance, a player would prefer that other players take the lead in link formation before plunging in herself. These phenomena have been noted in other contexts (see [7,1]). Fortunately, we are able to show in Theorem 4 that the complete graph (which must be socially efficient) will be the unique absorbing limit of the network formation process for some equilibrium profile.

Of course, "static" coordination failures can arise even in our dynamic framework. We provide a particularly stark example of this in Example 3, where we show that all matched pairs may break *all* links at the complete graph *even when it is the unique socially efficient network*. An implication of such static coordination failures is that typically efficiency cannot be sustained at all equilibria.

2. Valuation structures

Let I be a finite index set of players, and g an *undirected* graph on I. Such a graph, or network, is formally just a collection of ij pairs, the interpretation being that i and j are "linked".[4] We use the notation $g + ij$ to denote the new graph obtained from g by linking i and j.

A *component* of a network g is a subset c of g such that no $i \in c$ is linked outside c and such that every distinct i and j in c are directly or indirectly linked.[5] Let $C(g)$ denote all the components of g. For each $c \in C(g)$, let $I(c)$ denote the set of individuals in c.

Let G denote the set of all graphs on all nonempty subsets of I. The complete network, denoted $\tilde{g}$, is the graph in which all individuals are linked to one another.

Given any graph g, and component c in $C(g)$, $w(c, g)$ is the *value* or total "worth" of players in c. The total value of g is

$$w(g) \equiv \sum_{c \in C(g)} w(c, g). \tag{1}$$

We will say that $\mathbf{w}$ is an *additive function* if the value of any component c is independent of the structure of links of players not in c. In this case, we may as well use the notation $w(c)$ instead of $w(c, g)$. For such functions we also normalize by setting the value of singleton components equal to zero: $w(\{i\}) = 0$ for all i.

Notice that an additive function $\mathbf{w}$ is a generalization of TU-characteristic functions in cooperative game theory. However, our more general formulation allows for *externalities* across components of a graph, and so represents a generalization of partition functions since the value of a component depends not only on the coalition structure as in partition functions, but also on *how* the players in c are linked to each other.

Let W be the set of all worth functions $\mathbf{w}$ defined on all (c, g) pairs, where g is a network and c a component of g.

2.1. Allocation rules

An *allocation rule* is a mapping $\mathbf{a} : G \times W \to \mathbb{R}^n$ uch that $\sum_{i \in I} a_i(g, \mathbf{w}) = w(g)$, for all worth functions $\mathbf{w}$ and graphs g. The rule specifies the (one-period) payoffs to each

[4] Because the graph is undirected, these links are reciprocal. For analyses of valuation structures which are directed graphs, see [3,11].

[5] Thus isolated singletons are components by definition.

player i for every conceivable network and worth function. We will refer to the pair $(\mathbf{a}, \mathbf{w})$ as a *valuation structure*.

An allocation rule satisfies *component balance* if for all $w \in W$, for all $g \in G$, and for all $c \in C(g)$, $\sum_{i \in I(c)} a_i(g, \mathbf{w}) = w(c, g)$. This restriction rules out any cross-subsidization across links.

Throughout the paper, we will assume that the allocation rule satisfies component balance.

An allocation rule is *anonymous* if it distributes payoffs that depend only on player position in the network, and the particular worth function, and not player labels. Formally, if π is a permutation of I, let c^π be the appropriate transformation of c for every component of g, and also define g^π in similar fashion. For any $\mathbf{w}$, define $\mathbf{w}^\pi$ by $w^\pi(c^\pi, g^\pi) = w(c, g)$. Then $\mathbf{a}$ is *anonymous relative to* $(g, \mathbf{w})$ if for any permutation π, $a_{\pi(i)}(g^\pi, \mathbf{w}^\pi) = a_i(g, \mathbf{w})$. Say that the rule is *anonymous* (without qualification) if it is anonymous relative to every $(g, \mathbf{w})$.

One rule which is both component balanced and anonymous is the *component-wise egalitarian* allocation rule. This rule distributes worth equally within each component of a graph. That is, letting $\mathbf{a}^{\mathrm{e}}$ denote the component-wise egalitarian rule, we have

$$\text{For all } i \in I, \quad a_i^{\mathrm{e}}(g, w) = \frac{w(c, g)}{|c|}, \text{ where } c \in C(g), \quad i \in I(c).$$

2.2. Efficiency

Given some valuation structure, one might consider different notions of (static) efficiency for networks. [6] For instance, efficiency could correspond to maximizing *aggregate* payoffs: a graph g is *strongly efficient* if $w(g) \geqslant w(g')$ for all $g' \in G$.

A more conservative definition would allow for limited transferability, so that the constraints inherent in a given allocation rule are taken into account. In this spirit, a graph g is (weakly) *efficient* relative to $(\mathbf{a}, \mathbf{w})$ if there is no other $g' \in G$ such that $a_i(g', \mathbf{w}) \geqslant a_i(g, \mathbf{w})$ for all $i \in I$ with strict inequality for some $j \in I$.

2.3. Some restrictions on valuation structures

Two specific valuation structures will play a role in what follows. First, a valuation structure $(\mathbf{a}, \mathbf{w})$ exhibits *link monotonicity* if for every network g and all $i, j \in I$, $a_i(g + ij, w) > a_i(g, w)$ and $a_j(g+ij, w) > a_j(g, w)$ whenever $ij \notin g$. That is, link monotonicity requires that an individual's payoff is increasing in the number of her own links.

To be sure, link monotonicity allows for the possibility that an individual's payoff may go down if *other* players set up bilateral links. Specifically, the complete network $\tilde{g}$ may not be efficient even when the network structure displays link monotonicity. The example below shows that when $|N| = 3$, the complete network may violate strong efficiency. More complicated examples can be constructed to illustrate the possible violation of (weak) efficiency when $|N| \geqslant 4$.

[6] See [19].

Example 1. Let $N = \{1, 2, 3\}$. $\mathbf{w}$ is additive and symmetric with $w(\{ij\}) = 2$, $w(\{ij, jk\}) = 7/4$, and $w(\tilde{g}) = 3/2$. Moreover, $a_i(\{ij\}, \mathbf{w}) = a_j(\{ij\}, \mathbf{w}) = 1$, $a_i(\{ij, jk\}, \mathbf{w}) = a_k(\{ij, jk\}, \mathbf{w}) = 1/4$, $a_j(\{ij, jk\}, \mathbf{w}) = 5/4$, and $a_l(\tilde{g}, \mathbf{w}) = 1/2$ for all $l \in N$. Obviously, link monotonicity is satisfied, but the complete network is inefficient.

A valuation structure $(\mathbf{a}, \mathbf{w})$ displays *increasing returns to link creation* (IRL) if

(i) $\mathbf{w}$ is additive and $w(\tilde{g}) > 0$ (with $w(\{i\})$ normalized to 0 for all i);

(ii) whenever c is a nonsingleton component of some g with $w(c) \geqslant 0$, then $w(c) < w(c')$ for all $c' \supset c$;

(iii) for components c as described in (ii), if $i \in I(c)$ but $ij \notin g$, then $a_k(g + ij, \mathbf{w}) > a_k(g, \mathbf{w})$ for $k = i, j$.

The formalities of the definition look complicated but the main idea is very simple. A valuation structure satisfies IRL if along every nested chain of "increasingly connected" networks, there is a threshold (nonsingleton) network for which the worth turns nonnegative, and both aggregate payoffs as well as payoffs of individuals who form extra links then increase as the network becomes even larger. The point is that between the "empty network" of singletons and the threshold(s) there may lie intermediate networks that generate negative values.

Of course, link monotonicity and IRL are different conditions. The former applies to all $\mathbf{w}$, not just to additive functions, while the latter is restricted to the additive case. At the same time, the latter condition only imposes link monotonicity on a subcollection of components, not everywhere, though it also requires that aggregate worth also increase over this subcollection. This last condition helps to guarantee that under IRL, the complete network is the unique strongly efficient network. In contrast, we have already described an example to show that $\tilde{g}$ may not be strongly efficient when the valuation structure satisfies link monotonicity.

3. Some examples

In this section, we provide some examples that illustrate our general framework.

3.1. Connections

This model is due to [21]. Links represent social relationships. Individuals i and j are "friends" if they are linked together, and friendship is valuable. Individuals also benefit from indirect relationships—a "friend of a friend" brings additional benefit, which deteriorates, however, in the "distance" of the relationship. Let $\pi < 1$ be the benefit that i gets from a direct link with j, π^2 the benefit that i gets from someone who is at a distance two, and so on. Then

$$a_i(g, \mathbf{w}) = \sum_{j \neq i} \pi^{t(ij)} - \#\{k : ik \in g\}d, \tag{2}$$

where $t(ij)$ is the number of links in the shortest path between i and j, and d is the cost per link that i has to pay for each direct link. Here, the total value of a network is simply $w(g) = \sum_{i \in I} a_i(g, w)$.

The nature of strongly efficient graphs depend upon the relative values of π and d. If $d < \pi - \pi^2$, then the complete graph $\tilde{g}$ is strongly efficient. In this case, the valuation structure satisfies both link monotonicity as well as IRL.

A star [7] encompassing all agents is the unique strongly efficient graph for intermediate values of d.

If $d > \pi + (\frac{N-2}{2})\pi^2$, then the empty graph is the unique strongly efficient graph.

3.2. Group insurance

Consider n identical farmers producing random outputs. Any farmer can have a “high” output (of one unit) with probability p, or a low output (of zero units) with the remaining probability. These probabilities are iid across farmers. Each farmer is risk-averse, with v being the common increasing, strictly concave utility function.

Any two farmers can be connected at a cost of d. Assume that any group of connected farmers can mutually insure each other; suppose that the insurance contract is such that each member of the group will get an equal share of the total realized endowment net of the costs of the links.

Let c be a connected community of farmers with cardinality k and overall connection costs equal to $d(c)$. Then

$$w(c) = k \sum_{l=0}^{k} p^l (1-p)^{k-l} \binom{k}{l} v\left(\frac{l-d(c)}{k}\right)$$

and

$$a_i(c, \mathbf{w}) = \frac{w(c)}{k}.$$

Of course, efficiency requires that each component be minimally connected as long as $d > 0$.

Notice that a_i is increasing in the size of the connected component as long as the connection cost d is small, but for any positive connection cost must ultimately decline if the total number of farmers is large enough.

3.3. Collaboration

This is due to [18]. [8] Consider an oligopoly setting where firms form pairwise collaborative links with other firms. The collaboration could involve joint research activities, sharing knowledge about markets, sharing facilities such as distribution channels. A link between firms i and j yields lower costs of production for the two firms. Any collaboration network thus induces a distribution of costs across firms. Given these costs, firms subsequently compete on the product market as Cournot oligopolists.

[7] A star is a graph with a central node to which every other node is connected, with no other links.

[8] See also [17].

Assume that all firms have constant marginal costs of production, given by c_i for firm *i*. Given a graph *g*, let $\mu_i(g)$ denote the number of firms with which *i* has collaboration links. Then the resulting marginal cost of firm *i* is

$$c_i(g) = c_i - \gamma\mu_i(g),$$

where $\gamma > 0$ is the cost reduction induced by each link.[9]

Suppose the inverse market demand curve is linear :

$$p = a - q.$$

The output produced by firm *i* in the Cournot game will be

$$q_i(g) = \frac{(a - nc_i + \sum_{j\neq i} c_j) + n\gamma\mu_i(g) - \gamma\sum_{j\neq i}\mu_j(g)}{n+1}$$

and its overall profit is $q_i(g)^2$. Notice that the profit of firm *i* increases if *i* sets up an additional link. It follows that the valuation structure satisfies link monotonicity. On the other hand, an additional link by two rival firms *k* and *l* reduces firm *i*'s profit. Total industry profit is not maximised when all firms form bilateral links, and so the valuation structure does not satisfy IRL.

4. Process of network formation

Suppose that at any date, a pair of players *i* and *j* is randomly chosen (with uniform probability) and endowed with the capacity to take actions at that date. Each of these players can *unilaterally* sever any existing link with any other player, and they can *bilaterally* form a link between the two of them if one doesn't exist to begin with. These actions create a (possibly) new graph, and then one-period payoffs are received according to the given allocation rule. The current period then ends, and the whole process begins again *ad infinitum*.

Thus there are two components of a strategy in force: unilateral, which involves link severance, and bilateral, which involves link creation. Throughout, we will assume that players follow Markov strategies; i.e., their actions will be presumed to depend only on the existing payoff-relevant state.

Because strategies involve some elements of correlation and independence, we will need to be more specific and careful in describing them. Suppose that two individuals "partially cooperate", as they do here in setting up a bilateral link, but also take independent actions, as they do here with link destruction. Then the bilateral creation of a link between *i* and *j* are commonly observed by the two players, and can therefore serve as correlation devices: either player can condition her unilateral actions on the joint decision to bring this link into existence. In contrast, unilateral link breaking cannot be conditioned upon (at least in the absence of an explicit sequential structure which we do not assume).

[9] Assume that γ is small, so that net marginal cost is always positive for each firm.

This suggests that the situation is formally equivalent to one in which (at any date) actions pertaining to the possible creation of an *ij* link are taken "first" and these are "followed" by the unilateral actions (by *i* and *j*) pertaining to all other existing links. [10] Let us make this approach more formal.

It will be useful to define a *principal state* as a collection $s = (g, ij)$, where *g* is the historically given graph and *ij* is the chosen active pair. Define an *intermediate state* as a collection $s = (g, ij, \zeta)$, where *g* and *ij* are as before, and ζ is a variable which takes the value 0 if the pair *ij* is not linked, and the value 1 if *ij* is linked. An intermediate state doesn't physically exist; it is a conceptual halfway point for defining unilateral actions; hence the choice of terminology. In contrast, a principal state physically exists at the start of a period. When there is no need for a distinction, we shall simply use "state" to denote either of the two varieties. Notice, too, that we use the same notation *s* which will also ease the writing.

For any intermediate state $s = (g, ij, \zeta)$, define $D_i(s) \equiv \{k|i$ and k are linked in that state$\}$, and likewise define $D_j(s)$. These are the sets of *existing* linkages to *i* and *j* which can be broken unilaterally. (By assumption, no links other than those pertaining to the active pair can be created during this period.)

Formally, then, (mixed) actions may be described as follows. At any principal state *s* with active pair *ij* it is simply a probability $\mu(s) = q$ of bilateral linkage between *i* and *j*. At any intermediate stage *s* with active pair *ij* it is a collection $\mu(s) \equiv \{v_i, v_j\}$, where for each $k = i, j$, v_k is a probability measure defined over all subsets (including the empty subsets) of $D_k(s)$. [11] We will let $\boldsymbol{\mu}$ stand for the entire profile of $\mu(s)$'s over all states (notice that $\mu(s)$ has a different interpretation depending on what sort of state we are looking at), and refer to $\boldsymbol{\mu}$ as a *strategy profile*.

A strategy profile precipitates—for each state *s*, principal or intermediate—some probability measure λ_s over the feasible set $F(s)$ of future networks starting from *s*. (We omit the tedious but entirely routine formulae that link the λ_s's to the underlying profile $\boldsymbol{\mu}$.) In particular, a Markov process is induced on the set *S* of principal states: at any principal state *s*, λ_s describes the movement to a new network, and the given random choice of active players moves the system to a new active pair.

The process creates values for each player. Assuming that the a_k's are vN-M payoffs, we can write—for every state *s* with active pair *ij*—the overall payoff to any person *k* (under the strategy profile $\boldsymbol{\mu}$) as the unique solution to the functional equation

$$V_k(s, \boldsymbol{\mu}) = \sum_{g' \in F(s)} \lambda_s(g')[a_k(g') + \delta_k \sum_{i'j'} \pi(i'j') V_k(s', \boldsymbol{\mu})],$$

where $\delta_k \in (0, 1)$ is the discount factor of agent *k*, λ_s is the probability over $F(s)$ associated with $\boldsymbol{\mu}$, $\pi(i'j')$ is the probability that a pair $i'j'$ will be active "tomorrow", and s' stands for the principal state $(g', i'j')$. (Note that V_k is well-defined on both principal and intermediate states.)

[10] The phrases that suggest chronology are deliberately in quotes because no real chronology is implied.

[11] As a matter of notation, we should also index the individual *v*-components by *s*, but this is notationally cumbersome and hopefully the context will prevent any confusion.

Finally, at the risk of minor notational abuse, we will find it convenient to use $V_k(g, \boldsymbol{\mu})$ to denote the (expected) payoff to k at a given network g, *before* the active pair at that network is selected. This is given by simply taking expectations over the choice of active pair:

$$V_k(g, \boldsymbol{\mu}) = \frac{2}{n(n-1)} \sum_{ij \in I \times I} V_k((g, ij), \boldsymbol{\mu}).$$

4.1. Equilibrium

Loosely speaking, an *equilibrium process of network formation* is a strategy profile $\boldsymbol{\mu}$ with the property that there is no active pair at any state s which can benefit—either unilaterally or bilaterally—by departing from $\mu(s)$. *The benefit is evaluated according to the value function introduced above*. The remainder of this section contains a precise formulation of this idea. Before the formalities are introduced, however, note the following points:

(1) Profitable deviations are *not* necessarily myopic: individuals take the ongoing process as given and evaluate the entire stream of consequences arising from a single action. One can imitate perfectly myopic behavior by taking the discount factor to zero, and perfect farsightedness by taking the opposite limit.

(2) Network formation and payoffs occur together. There is no “waiting” in the model until some “stable” network is formed, following which payoffs are assigned. Indeed, our definition permits cycles and continued flux in the network, and there is no difficulty at all in evaluating overall payoffs.

Now for a precise account. Fix some ongoing strategy profile $\boldsymbol{\mu}$ and an intermediate state s with active pair ij. A *unilateral move* for i at s (to be sometimes referred to as an i-unilateral move at s when it’s necessary to keep track of the relevant agent) is simply a collection $\mu'(s) = \{v_i', v_j\}$. That is, the ith component of $\mu(s)$ has (possibly) been altered from v_i to v_i'. Given a principal state s, a *bilateral move* for the active pair ij is simply a probability $\mu'(s)$ of ij-linkage.

A strategy profile $\boldsymbol{\mu}$ “perturbed” by an unilateral or bilateral move at s is still a strategy profile. We will occasionally use the notation $\boldsymbol{\mu}'$ to denote the new profile (the context will make clear exactly which move is generating $\boldsymbol{\mu}'$).

For an intermediate state s with active pair ij, and for some $k = i, j$, say that a k-unilateral move $\mu'(s)$ is *profitable* if

$$V_k(s, \boldsymbol{\mu}') > V_k(s, \boldsymbol{\mu}), \tag{3}$$

where $\boldsymbol{\mu}'$ is the strategy profile “induced” by the k-unilateral move $\mu'(s)$ (see previous paragraph). Likewise, for a principal state s with active pair ij, say that a bilateral move $\mu'(s)$ is *profitable* if

$$V_i(s, \boldsymbol{\mu}') > V_i(s, \boldsymbol{\mu}) \quad \text{and } V_j(s, \boldsymbol{\mu}') > V_j(s, \boldsymbol{\mu}), \tag{4}$$

where, again, $\boldsymbol{\mu}'$ is the strategy profile “induced” from $\boldsymbol{\mu}$ by the bilateral move $\mu'(s)$. A strategy profile $\boldsymbol{\mu}$ is an *equilibrium* if at no s is a unilateral or bilateral move profitable.

Notice how our description of equilibrium subsumes a “perfection” requirement. An equilibrium must be immune to *all* profitable moves, including those starting from principal or intermediate states that may never be reached. [12]

4.2. Existence

One can establish the following.

Theorem 1. *An equilibrium in mixed bilateral and unilateral strategies always exists.*

Proof. For every state s look at the space $U(s)$ of all possible $\mu(s)$. Let $\mathbf{U} \equiv \prod_{s \in S} U(s)$. (Note: with the obvious product topology, $\mathbf{U}$ is viewable as a compact, convex subset of some finite-dimensional Euclidean space.) For each s, we construct a nonempty-valued, convex-valued uhc correspondence Ψ_s from $\mathbf{U}$ to $U(s)$ in the following way.

Fix some $\boldsymbol{\mu} \in \mathbf{U}$, and consider any state s. If s is an intermediate state with active pair ij, maximize—for each $k \in \{ij\}$—the value of $V_k(s, \boldsymbol{\mu}')$ over all $\boldsymbol{\mu}'$ induced from $\boldsymbol{\mu}$ by k-unilateral moves at s. Gather all the (mixed) k-unilateral moves v'_k that achieve this maximum. Beue *kk* that the $\{ij\}$ pair is active? And if they do, can they condition their actions on the formation of the $\{ij\}$ link? Might some of these individuals be aware of these matters, and others not?

Fortunately, our equilibrium concept is robust enough to accommodate these alterations, though in the interest of focus we do not pursue the variants in this paper. For instance, consider the scenario in which third parties are free to sever links (in addition to the active pair), and they know the identity of the active pair. Because no commitment is assumed, suppose that third parties must move simultaneously against one another (and against the active pair). Formally, this amounts to having the third parties move at the *principal* state, while the members of the active pair continue to move in the intermediate stage (with effective knowledge of their own bilateral actions).

It is easy to see that the existence argument goes through with only minor changes.

The same is also true when there are several pairs of active players, provided that different pairs do not have players in common.

Potential problems might arise when active pairs “intersect”, especially if several pairs

5.1. Possibility of inefficiency

In order to demonstrate that the issue of sustaining efficiency in this framework is not a trivial proposition, we show that there are valuation structures in which *no* equilibrium strategy profile yields paths that are absorbed solely into a set of efficient networks. This can be viewed as the dynamic counterpart of the conflict between (static) stability and efficiency demonstrated by [21]. To show this, say that an allocation rule *permits limited transfers* if $a_i(g, w) \leqslant w(g)$ for all i whenever $w(g) \geqslant 0$.

An allocation rule which permits limited transfers does not allow other individuals to "overcompensate" any individual.

Suppose the allocation rule is anonymous and satisfies Limited Transfers. We construct below a (symmetric) worth function on three players such that the efficient network is not strongly absorbing at any pure strategy equilibrium. The worth function is such that the complete graph has a value 3α, while each one-link graph has value 2α. All other graphs have value 0. Then, the complete graph is the unique efficient network. Given the restrictions on the allocation rule, each player gets α if the complete graph forms, while players i and j also get α if they form the one-link graph $\{ij\}$. Then, once a graph $\{ij\}$ has formed, i and j have no incentive to move towards the complete graph since their payoffs at $\{ij\}$ and the complete graph are identical, *and* there will be some intermediate stages where they get zero. The proof below shows that the possibility of "coordination failures" does not cause the process to converge to the complete graph at *any* equilibrium.

Theorem 2. *Suppose that* **a** *is anonymous and permits limited transfers. Then there is* **w** *and* $\bar{\delta} < 1$ *such that for all* $\delta \in (\bar{\delta}, 1)$ *every pure strategy equilibrium profile generates paths that fail to exit the set of inefficient networks*.

Proof. Let $I = \{1, 2, 3\}$. Choose symmetric additive **w** such that $w(\{i\}) = 0$, $w(\{ij\}) =$ 2α, $w(\{ij, ik, jk$

(F2) requires that both i and j move away from $\{ij\}$ when they meet k. This is true for the following reason. *Some* player must move away from $\{ij\}$ since the complete network is strongly absorbing. Suppose only i either forms the one link network $\{ik\}$ or the two link network $\{ij, ik\}$ when the current network is $\{ij\}$. Then, i's payoff must be zero in all periods when the two-link networks are in place. (There must be *some* periods when the two-link networks are in place since the process converges to the complete network by assumption.) However, if i remains at $\{ij\}$—which he can do by a unilateral devido by a unilateral devido by zc,TS&j

Then,

$$V_1(\{13\}, 12) = \alpha + \delta V_1(\{12\}),$$

where

$$V_1(\{12\}) = \frac{a}{3-2\delta} + \frac{\delta V_1(g)}{3-2\delta}.$$

Suppose 1 deviates from μ^* at the principal state ({13}, 12), by retaining the link with 3 and refusing to form the link with 2. Denoting the resulting discounted payoffs by V',

$$V_1'((\{13\}, 12) = \alpha + \delta V_1'(\{13\}).$$

But

$$V_1'(\{13\}) = \frac{2\alpha}{3-2\delta} + \frac{\delta}{3-2\delta} V_1(\{13, 23\}) = \frac{2\alpha}{3-2\delta} + \frac{\delta\alpha}{(3-2\delta)^2(1-\delta)} > V_1(\{12\}).$$

Hence, μ^* cannot be an equilibrium in this case.

Case 2(b): *Suppose* $\{13\} \rightarrow_{12} \{12, 13\}$ *and* $\{13\} \rightarrow_{23} \{23\}$

In this case, 3 has a profitable unilateral deviation at ({13}, 23)-3 can retain link with 1 and refuse to form link with 2.

Case 2(c): *Suppose* $\{13\} \rightarrow_{12} \{12\}$ *and* $\{13\} \rightarrow_{23} \{23\}$.

Then,

$$V_3(\{13\}, 23) = \alpha + \delta V_3(\{23\}).$$

Also,

$$V_3(\{23\}) = \frac{\alpha}{3-\delta} + \frac{\delta}{3-\delta} V_3(\{13, 23\}) + \frac{\delta}{3-\delta} V_3(\{12\}),$$

$$V_3(\{12\}) = \frac{\alpha}{3-\delta} + \frac{\delta}{3-\delta} V_3(\{23\}) + \frac{\delta}{3-\delta} V_3(\{12, 13\}).$$

Using the fact that $V_3(\{12, 13\}) = V_3(\{13, 23\}) = \frac{\delta\alpha}{(3-2\delta)(1-\delta)}$, and simplifying,

$$V_3(\{23\}) = \frac{\alpha}{3-2\delta} + \frac{\delta}{3-2\delta} V_3(\{13, 23\}).$$

Hence,

$$V_3(\{13\}, 23) = \frac{3\alpha - \alpha\delta}{3-2\delta} + \frac{\delta^2}{3-2\delta} V_3(\{13, 23\}).$$

Now, suppose 3 deviates at the *intermediate* state so that after forming a link with 2, 3 retains the link with 1. Then,

$$V_3'(\{13\}, 23) = \delta V_3(\{13, 23\}).$$

Hence,

$$V_3'(\{13\}, 23) - V_3(\{13\}, 23) = \frac{3\alpha\delta}{(3-2\delta)^2} - \frac{3\alpha - \alpha\delta}{3-2\delta}.$$

This is positive for δ large enough.

Also, note that 2 is better off forming the link with 3 rather than remaining at {13}, even if 3 refuses to cut the link with 1. Hence, μ^* cannot be an equilibrium in this case either.

Using (F2), this exhausts all possible cases, and so establishes the theorem. □

Notice that the complete graph may be absorbing at *some* equilibrium. However, if the process of network formation "starts" at the empty network, then the complete graph will never be reached at any equilibrium—only one-link graphs will form. So, this example illustrates the importance of efficient graphs being sustained as strongly absorbing graphs.

At first sight, this type of failure to sustain an efficient network may appear similar to that arising in strictly superadditive (transferable utility) games with empty cores. Here too, the grand coalition may not form since some subset may do better on its own. However, this argument implicitly presumes that the members of the blocking coalition agree to "leave the game" and form a sub-society of their own. In other words, members of the blocking coalition assume that there is some commitment device which "binds" them together. In contrast, the current framework assumes very limited cooperation amongst individuals. So, when the network $\{ij\}$ forms, both *i* and *j* may anticipate that the other will form a link with *k*. Even though this does not bring any additional benefit to either *i* or *j*, these anticipations can in principle sustain each other. The theorem essentially demonstrates that this cannot happen in the specific example used in the proof.

5.2. Absorption into the complete graph

In this subsection, we both simplify and extend the logic of inefficient outcomes. The simplification is that we select the equilibrium in question (Theorem 2 applied to all equilibria). But we extend the argument in the sense that we provide a set of conditions (not just an example) under which the complete network is strongly absorbing (for some equilibrium). Note that this says nothing about efficiency (after all, the complete network may be inefficient).

Specifically, we now show that if the valuation structure satisfies link monotonicity, then the complete graph $\tilde{g}$ can be supported as a strongly absorbing graph at *some* equilibrium strategy profile. (However, we also remark that the complete graph is not necessarily strongly absorbing at *all* equilibria.)

Theorem 3. *Suppose* $(\mathbf{a}, \mathbf{w})$ *satisfies link monotonicity. Then, for all* $\delta \in (0, 1)$, *there is some equilibrium* μ^* *such that* $\tilde{g}$ *is strongly absorbing.*

Proof. Consider the strategy profile μ^* where at any principal state (g, ij), *i* and *j* form the link *ij* (if unlinked), and at every intermediate state, no link is severed. We show that such μ^* is an equilibrium strategy profile.

It will be sufficient to show that

$$\text{For all } g, \text{ for all } ij \notin g, V_i(g+ij, \mu^*) > V(g, \mu^*). \tag{6}$$

We prove that (6) is true by induction on the number of links ("distance") that separates g from $\tilde{g}$.

If g and $\tilde{g}$ are separated by a single link, then $g = \tilde{g} - ij$. In this case, given the strategies of all players, i and j obtain (in each period) precisely $a_i(g, \mathbf{w})$ and $a_j(g, \mathbf{w})$ as long as they do not form a link, and $a_i(\tilde{g}, \mathbf{w})$ and $a_j(\tilde{g}, \mathbf{w})$ if they do. So given link monotonicity, (6) is trivially true in this case.

Next, define $M \equiv \binom{n}{2}$. This is the number of all possible pairs, and therefore also the maximal "distance" between $\tilde{g}$ and any g. Suppose, inductively, that for $2 \leqslant K \leqslant M$, (6) holds for all g which are at a distance of $K-1$ or less from $\tilde{g}$. Pick any g (with $ij \notin g$) at distance K from $\tilde{g}$. Define $g' \equiv g + ij$. Then

$$\begin{aligned}
&V_i(g, \mu^*) \\
&= \frac{M-K}{M}\big(a_i(g, \mathbf{w}) + \delta V_i(g, \mu^*)\big) + \frac{1}{M}\left(\sum_{kl \notin g}\big(a_i(g+kl, \mathbf{w}) + \delta V_i(g+kl, \mu^*)\big)\right) \\
&= \frac{(M-K)a_i(g, \mathbf{w}) + a_i(g', \mathbf{w}) + \delta V_i(g', \mu^*) + \sum_{kl \notin g'}(a_i(g+kl, \mathbf{w}) + \delta V_i(g+kl, \mu^*))}{M - \delta(M-K)}.
\end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}
V_i(g', \mu^*) &= \frac{(M-K+1)a_i(g', \mathbf{w}) + \sum_{kl \notin g'}\big(a_i(g'+kl, \mathbf{w}) + \delta V_i(g'+kl, \mu^*)\big)}{M - \delta(M-K+1)} \\
&> \frac{(M-K+1)a_i(g', \mathbf{w}) + \sum_{kl \notin g'}(a_i(g+kl, \mathbf{w}) + \delta V_i(g+kl, \mu^*))}{M - \delta(M-K+1)},
\end{aligned} \tag{8}$$

where the inequality invokes both link monotonicity and the induction hypothesis (noting that for all $kl \notin g'$, $g' + kl = \{g + kl\} + ij$, and that $g + kl$ is at a distance of $K - 1$ from $\tilde{g}$).

Combining (7) and (8), we may conclude that

$$\begin{aligned}
&[M - \delta(M-K+1)]V_i(g', \mu^*) - [M - \delta(M-K)]V_i(g, \mu^*) \\
&\quad > (M-k)[a_i(g', \mathbf{w}) - a_i(g, \mathbf{w})] - \delta V_i(g', \mu^*),
\end{aligned}$$

so that

$$V_i(g', \mu^*) - V_i(g, \mu^*) > \frac{M-K}{M - \delta(M-K)}[a_i(g', \mathbf{w}) - a_i(g, \mathbf{w})] > 0,$$

the last inequality following from link monotonicity once again. This completes the inductive step. □

Link monotonicity does not imply that the complete graph is strongly absorbing at *all* equilibria. In Example 1, it is easy to check that there can be an equilibrium in which the one-link graph is an absorbing graph.

Notice, too, that the variations on the solution concept described in Section 4.3 do not affect Theorem 3. For instance, if all players can delete links (and not just the active pair), the equilibrium described above survives, and displays exactly the same properties.

5.3. A positive result on efficiency

Finally, we turn to a positive result regarding efficiency. We show that if the valuation structure satisfies IRL, and if the allocation rule is the component-wise egalitarian rule, then the complete graph will be strongly absorbing at some pure strategy equilibrium profile. A first reaction may be that this is an obvious result. After all, if $\tilde{g}$ is the unique strongly efficient graph, then surely everyone has a common interest in reaching $\tilde{g}$ and then staying there? However, suppose that aggregate payoffs (normalized so that isolated individuals obtain 0) are negative for "small" graphs. Then, all individuals prefer to join the network *after* it has reached the critical threshold beyond which payoffs are nonnegative. Indeed, it is a nontrivial issue to show that free-riding behavior does not become so pervasive so as to altogether negate the convergence of the process to the complete network.

The following example illustrates the nature of the free-riding behavior when the valuation structure satisfies IRL.

Example 2. Let $N = \{1, 2, 3\}$, $w(g) = -4$ if $\#g = 1$, $w(\{ij, jk\}) = -3$, $w(\tilde{g}) = 3$. Suppose the allocation rule is the component-wise egalitarian rule.

Then, it cannot be an equilibrium for all pairs to form a link at all networks. For suppose, both 1 and 2 want to form additional links at each opportunity. This then permits 3 to free-ride. To check this, let V_3 denote discounted payoffs if 3 also agrees to form a link at each opportunity. Routine calculations yield (for $i = 1, 2$)

$$V_3(\emptyset, 3i) = -\frac{-6}{3 -}$$

Proof. If a component c is nonsingleton and $w(c) \geqslant 0$, call it *nonnegative*. Define a strategy profile μ^* as follows. Consider any state $s = (g, ij)$.

(I) If either i or j (or both) are members of some nonnegative component of g (perhaps different ones), then i and j retain all existing links and form the link ij if it did not exist before (and if s is a principal state).

(II) Otherwise, i and j follow any equilibrium strategy profile in the "restricted" game where strategies follow (I) whenever (I) applies. [16]

Notice that in Case I, the immediate payoffs of both i and j must go up (this is easy to verify using the definition of IRL). This means that link monotonicity is satisfied on the subdomain in which (I) applies. Because w is additive, the behavior specified by (I) is an equilibrium, by Theorem 3. So μ^* is an equilibrium in the overall game.

We will now show that the equilibrium entails convergence to $\tilde{g}$. To this end, we first claim

Fact 1. *If $V_i(s, \mu^*) > 0$ for any state s and any i, the process must converge to $\tilde{g}$ from that state.*

To prove this, let H be the set of all graphs that contain at least one nonnegative component. By (I), if the process enters H, then it *must* converge to $\tilde{g}$ a.s. But if the process does not ever enter H, then no player can ever earn a strictly positive payoff, by the definition of IRL. This proves Fact 1.

Fact 2. *There exists a $\bar{\delta} \in (0, 1)$ such that for all $\delta \geqslant \bar{\delta}$ and for any graph g, there is some stage of the form $s = (g, ij)$ where the active players i and j earn a positive payoff.*

To show this, first note that once the process enters H, there is a stochastic, bounded time (independent of δ) within which the complete graph $\tilde{g}$ will be reached. Notice that $a_i(\tilde{g}, \mathbf{w}) > 0$ for all i, so for any i,

$$V_i(s, \mu^*) \geqslant \underline{V}_i($$

Now, take any g' and *kl* such that $g + kl = \bar{g}$, and consider $s' = (g, kl)$. Since *kl* can form the link *kl*, for each $i \in \{k, l\}$

$$V_i(s', \mu) \geqslant a_i(\bar{g}, w) + \frac{q}{M}\delta^2 \underline{V}_i(\delta) - \frac{M-q}{M}L.$$

Notice that for sufficiently large values of δ, $V_i(s', \mu^*) \to \infty$ as $\delta \to 1$. Therefore the active pair (k, l) enjoys a strictly positive payoff at this stage.

Continuing these arguments inductively, it is possible to establish Fact 2 for all initial networks g.

Combining Facts 1 and 2, the proof of the proposition is complete. □

Our last example shows that "static" coordination failures can occur even when the valuation structure satisfies IRL. The example shows that $\tilde{g}$ is not strongly absorbing under some equilibrium profiles even though IRL is satisfied.

Example 3. Suppose $N = \{1, 2, 3, 4\}$, the valuation structure satisfies IRL, with $w(\tilde{g}) = 4$, $w(\{C$

In addition to the use of this dynamic framework, which we borrow from [23], we posit a limited form of cooperation that is grounded firmly in the networks literature: links are formed bilaterally and destroyed unilaterally, as in [21]. Thus, instead of arbitrary coalitions being active at any date, an active *pair* is randomly chosen, and this pair can form a bilateral link if one does not exist between them. (They can also sever links unilaterally.) This yields a solution concept that we show to be nonempty in a wide class of situations. We then apply the concept to the study of efficiency. In particular, we show that there are valuation structures in which no equilibrium strategy profile can sustain efficient networks. We then provide sufficient conditions under which the equilibrium process will yield efficient outcomes.

Acknowledgments

Thanks to Guillaume Haeringer and an anonymous referee for helpful comments.

References

[1] A. Adserà, D. Ray, History and coordination failure, J. Econ. Growth 3 (1998) 267–276.
[2] R. Aumann, R. Myerson, Endogenous formation of links between players and coalitions: an application of the shapley value, in: A. Roth (Ed.), The Shapley Value, Cambridge University Press, Cambridge, 1988, pp. 175–191.
[3] V. Bala, S. Goyal, Self-organization in communication networks, Econometrica 68 (2001) 1181–1230.
[4] O. Bandiera, I. Rasul, Social networks and the adoption of new technology in Northern Mozambique, mimeo., London School of Economics, CEPR Discussion Paper 3341, 2002.
[5] Y. Bramoullé, R. Kranton, Social learning, social networks, and search, mimeo., Department of Economics, University of Maryland, 2002.
[6] A. Calvo-Armengo, M. Jackson, Social networks and the resulting dynamics and patterns of employment and wages, mimeo., CalTech., 2001.
[7] C. Chamley, D. Gale, Information revelation and strategic delay in a model of investment, Econometrica 62 (1994) 1065–1085.
[8] M. Chwe, Farsighted coalitional stability, J. Econ. Theory 63 (1994) 299–325.
[9] T. Conley, C. Udry, Learning about a new technology: pineapple in Ghana, mimeo., Department of Economics, Yale University, 2002.
[10] S. Currarini, M. Morelli, Network formation with sequential demands, Rev. Econ. Design 5 (2000) 229–250.
[11] B. Dutta, M.O. Jackson, The stability and efficiency of directed communication networks, Rev. Econ. Design 5 (2000) 251–272.
[12] B. Dutta, M.O. Jackson (Eds.), Models of the Formation of Networks and Groups, Springer, Berlin, 2003.
[13] B. Dutta, S. Mutuswami, Stable networks, J. Econ. Theory 76 (1997) 322–344.
[14] B. Dutta, A. van den Nouweland, S. Tijs, Link formation in cooperative situations, Int. J. Game Theory 27 (1998) 245–256.
[15] M. Fafchamps, S. Lund, Risk-sharing networks in rural Philippines, mimeo., Stanford University, 2000.
[16] G. Genicot, D. Ray, Group formation in risk-sharing arrangements, Rev. Econ. Stud. 70 (2003) 87–113.
[17] S. Goyal, J. Moraga-Gonzalez, R & D networks, Rand J. Econ. 32 (2001) 686–707.
[18] S. Goyal, S. Joshi, Networks of collaboration in oligopolies, Games Econ. Behav. 43 (2003) 57–85.
[19] M.O. Jackson, The stability and efficiency of economic and social networks, in: M. Sertel, S. Koray (Eds.), Adv. Econ. Design, Springer-Verlag, Heidelberg, 2003.
[20] M.O. Jackson, A. Watts, The evolution of social and economic networks, J. Econ. Theory 106 (2002) 265–291.
[21] M.O. Jackson, A. Wolinsky, A strategic model of social and economic networks, J. Econ. Theory 71 (1996) 44–74.

[22] S. Kariv, Social learning in a network, mimeo., Department of Economics, New York University, 2002.
[23] H. Konishi, D. Ray, Coalition formation as a dynamic process, J. Econ. Theory 110 (2003) 1–41.
[24] R. Kranton, D. Minehart, Competition for goods in buyer–seller networks, Rev. Econ. Design 5 (2000) 301–332.
[25] R. Kranton, D. Minehart, A theory of buyer–seller networks, Amer. Econ. Rev. 2001.
[26] R. Myerson, Graphs and cooperation in games, Math. Oper. Res. 2 (1977) 225–229.
[27] F.H. Page, M.H. Wooders, S. Kamat, Networks and farsighted stability, mimeo., Working Paper 621, Department of Economics, University of Warwick, 2001.
[28] M. Slikker, A. van den Nouweland, Network formation models with costs for establishing links, Rev. Econ. Design 5 (2000) 333–362.
[29] M. Slikker, A. van den Nouweland, A one-stage model of link formation and payoff division, Games Econ. Behav. 34 (2001) 153–175.
[30] L. Tesfatsion, A trade network game with endogenous partner selection, in: H. Amman et al. (Eds.), Computational Approaches to Economic Problems, Kluwer Academic Publishers, Dordrecht, 1997, pp. 249–269.
[31] L. Tesfatsion, Gale-shapley matching in an evolutionary trade network game, Iowa State University Economic Report no. 43, 1998.
[32] P. Wang, A. Watts, Formation of buyer–seller trade networks in a quality-differentiated product market, mimeo., Southern Illinois University, 2002.
[33] S. Wasserman, K. Faust, Social Network Analysis: Methods and Applications, Cambridge University Press, Cambridge, 1994.
[34] A. Watts, A dynamic model of network formation, Games Econ. Behav. 34 (2001) 331–341.
[35] G. Weisbuch, A. Kirman, D. Herreiner, Market organisation, mimeo Ecole Normal Superieure, 1995.